

# LIMITS OF TEICHMÜLLER GEODESICS IN THE UNIVERSAL TEICHMÜLLER SPACE

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ABSTRACT. Thurston's boundary to the universal Teichmüller space  $T(\mathbb{H})$  is the set of asymptotic rays to the embedding of  $T(\mathbb{H})$  in the space of geodesic currents; the boundary is identified with the projective bounded measured laminations  $PML_{bdd}(\mathbb{H})$  of  $\mathbb{H}$ . We prove that each Teichmüller geodesic ray in  $T(\mathbb{H})$  has a unique limit point in Thurston's boundary to  $T(\mathbb{H})$  unlike in the case of closed surfaces.

## 1. INTRODUCTION

The Teichmüller space  $T(\mathbb{D})$  of the unit disk  $\mathbb{D}$ , called the *universal* Teichmüller space, consists of all quasiconformal maps  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  which fix 1,  $i$  and  $-1$  on the unit circle  $\mathbb{S}^1$  (cf. [4]). The Teichmüller space of an arbitrary hyperbolic surface embeds in  $T(\mathbb{D})$  as a complex Banach submanifold. Thurston's boundary to the universal Teichmüller space  $T(\mathbb{D})$  is the space of projective bounded measured laminations  $PML_{bdd}(\mathbb{D})$  of  $\mathbb{D}$  (cf. [18], [20]). We study the limits of Teichmüller geodesic rays on Thurston's boundary to  $T(\mathbb{D})$ .

Bonahon [2] defined an embedding of the Teichmüller space  $T(S)$  of a closed surface  $S$  of genus at least two into the space of geodesic currents (equipped with the weak\* topology). The space of asymptotic rays to the image of the embedding of  $T(S)$  is identified with the space of projective measured lamination of  $S$ -Thurston's boundary to  $T(S)$ . The universal Teichmüller space  $T(\mathbb{D})$  embeds into the space of geodesic currents of  $\mathbb{D}$  when geodesic currents are equipped with the *uniform* weak\* topology (cf. [18], [14], [17]) and this embedding is real analytic (cf. Otal [15]). The image of  $T(\mathbb{D})$  in the space of geodesic currents is closed and unbounded, and the space of asymptotic rays to the image of  $T(\mathbb{D})$ -Thurston's boundary to  $T(\mathbb{D})$ - is identified with the projective bounded measured laminations  $PML_{bdd}(\mathbb{D})$  (cf. [18], [17]). In particular, the earthquake paths  $t \mapsto E^{t\mu}|_{\mathbb{S}^1}$  as  $t \rightarrow \infty$  accumulate to their corresponding projective earthquake measures  $[\mu] \in PML_{bdd}(\mathbb{D})$  in the uniform weak\* topology (cf. [18], [17]). The construction of Thurston's boundary works for all hyperbolic surfaces simultaneously since any invariance under a Fuchsian group is preserved under the construction.

In the case of closed surfaces, Masur [13] proved that the Teichmüller geodesic rays obtained by shrinking the vertical trajectories of holomorphic quadratic differentials with uniquely ergodic vertical foliations converge to the projective classes of their vertical foliations in Thurston's boundary. On the other hand, if the vertical foliation of a holomorphic quadratic differential consists of finitely many cylinders then the limit of the Teichmüller geodesic on Thurston's boundary is the projective class of the measured lamination supported on finitely many simple closed geodesics homotopic to the cylinders with equal weights (cf. [13]). In both cases the Teichmüller geodesic rays have unique endpoints on Thurston's boundary and the endpoints depend only on the vertical foliations. However, when the vertical foliations of holomorphic quadratic differentials on closed surfaces are not uniquely ergodic then the limit sets of the corresponding Teichmüller rays consist of more than one point (cf. Lenzhen [12], Leininger-Lenzhen-Rafi [11], and Chaika-Masur-Wolf [3]).

We consider the limits of Teichmüller geodesic rays in the universal Teichmüller space  $T(\mathbb{D})$  corresponding to integrable holomorphic quadratic differentials. In our previous work we showed that when a holomorphic quadratic differential has no zeroes in  $\mathbb{D}$ , and the natural parameter maps the unit disk onto a domain in  $\mathbb{C}$  between the graphs of two functions, then the Teichmüller geodesic ray has a unique endpoint on Thurston's

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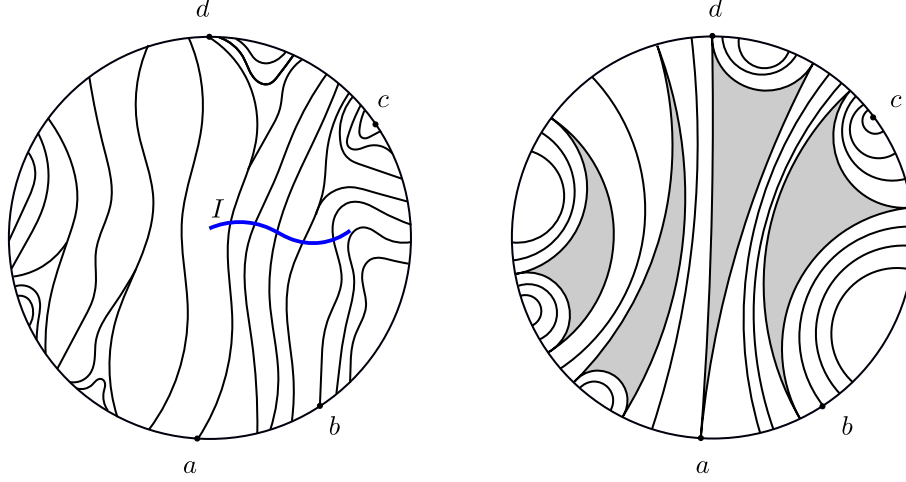


FIGURE 1. The vertical foliation of  $\varphi$  and the corresponding geodesic lamination  $v_\varphi$ . The measure  $\mu_\varphi([a, b] \times [c, d])$  is obtained by integration  $\int_I \frac{1}{l_\varphi(z)} |\sqrt{\varphi(z)} dz|$ .

boundary of  $T(\mathbb{D})$ , but the endpoint depends on both vertical and horizontal foliations of  $\varphi$  (cf. [7]). We extend this result to all integrable holomorphic quadratic differentials on the unit disk  $\mathbb{D}$ .

The hyperbolic plane is identified with the unit disk  $\mathbb{D}$  and the visual boundary of the hyperbolic plane is identified with the unit circle  $\mathbb{S}^1$ . A (hyperbolic) geodesic in  $\mathbb{D}$  is uniquely determined by its endpoints; the space of geodesics of  $\mathbb{D}$  is identified with  $\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$ . Let  $\varphi$  be an arbitrary integrable holomorphic quadratic differential on the unit disk  $\mathbb{D}$ . Each vertical trajectory of  $\varphi$  has two distinct endpoints on the boundary circle  $\mathbb{S}^1$  of the unit disk  $\mathbb{D}$  (cf. [21]). Thus each vertical trajectory of  $\varphi$  is homotopic to a unique geodesic of  $\mathbb{D}$  relative ideal endpoints on  $\mathbb{S}^1$ . Let  $v_\varphi$  be the set of the geodesics in  $\mathbb{D}$  homotopic to the vertical trajectories of  $\varphi$  (cf. Figure 1). Given a box of geodesics  $[a, b] \times [c, d] \subset \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$ , denote by  $I_{[a, b] \times [c, d]}$  any (at most countable) union of sub-arcs of horizontal trajectories that intersects exactly once each vertical trajectory of  $\varphi$  with one endpoint in  $[a, b]$  and the other endpoint in  $[c, d]$ , and that does not intersect any other vertical trajectories of  $\varphi$ .

We define a measured lamination  $\mu_\varphi$  of  $\mathbb{D}$  supported on  $v_\varphi$  by

$$(1) \quad \mu_\varphi([a, b] \times [c, d]) = \int_{I_{[a, b] \times [c, d]}} \frac{1}{l(x)} dx,$$

where  $l(x)$  is the  $\varphi$ -length (i.e. the length induced by  $\int |\sqrt{\varphi(z)} dz|$ ) of a vertical trajectory through  $x \in I_{[a, b] \times [c, d]}$  and the integration is in the natural parameter of  $\varphi$ . We obtain (cf. Proposition 4.4 and proof of Theorem 4.5)

**Proposition 1.** *Let  $\mu_\varphi$  be the measured lamination homotopic to the vertical foliation of an integrable holomorphic quadratic differential  $\varphi$  on  $\mathbb{D}$  defined by the above integration. Then*

$$\|\mu_\varphi\|_{Th} = \sup_{[a, b] \times [c, d]} \mu_\varphi([a, b] \times [c, d]) < \infty$$

where the supremum is over all boxes of geodesics  $[a, b] \times [c, d] \subset \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$  with  $cr(a, b, c, d) = 2$ .

The measured lamination  $\mu_\varphi$  satisfies

$$\mu_\varphi(\{a\} \times [c, d]) = 0$$

for all  $a \in \mathbb{S}^1$  and  $[c, d] \subset \mathbb{S}^1$ , and in particular  $\mu_\varphi$  does not have atoms.

For  $\epsilon > 0$ , let  $T_\epsilon$  be the Teichmüller mapping that shrinks the vertical trajectories of  $\varphi$  by a multiplicative constant  $\epsilon$ . The Teichmüller geodesic ray  $\epsilon \mapsto T_\epsilon$  as  $\epsilon \rightarrow 0^+$  leaves every bounded subset of the universal Teichmüller space  $T(\mathbb{D})$ . We obtain (cf. Theorem 4.5)

**Theorem 1.** *Let*

$$\epsilon \mapsto T_\epsilon$$

*be the Teichmüller geodesic ray in  $T(\mathbb{D})$  that shrinks the vertical trajectories of an integrable holomorphic quadratic differential  $\varphi$  by a multiplicative constant  $\epsilon > 0$ . Then*

$$T_\epsilon \rightarrow [\mu_\varphi] \in PML_{bdd}(\mathbb{D})$$

*as  $\epsilon \rightarrow 0^+$  in Thurston's closure  $T(\mathbb{D}) \cup PML_{bdd}(\mathbb{D})$  of  $T(\mathbb{D})$ , where  $\mu_\varphi$  is the measured lamination defined by equation (1) and the convergence is in the weak\* topology on geodesic currents.*

*In particular, the limit set of any Teichmüller ray in  $T(\mathbb{D})$  consists of a unique point.*

**Remark 1.** The limit point  $\mu_\varphi$  depends on the vertical foliation and on the lengths of the vertical trajectories unlike for closed surfaces. The lengths of vertical trajectories are given by the transverse measure to the horizontal foliation. Therefore the limit point depends on both vertical and horizontal foliations of  $\varphi$  which is a new phenomenon that does not appear for closed surfaces.

**Remark 2.** The measure  $\mu_\varphi([a, b] \times [c, d])$  is in fact the conformal modulus of the family of vertical trajectories of  $\varphi$  with one endpoint in  $[a, b]$  and another endpoint in  $[c, d]$  (cf. Proposition 4.3).

**Remark 3.** The above theorem is motivated by the results of Masur [13] in the case of a closed surface. The major difference in this work is that the hyperbolic plane has no closed geodesics and that the universal Teichmüller space is infinite dimensional non-separable Banach manifold. The convergence questions that arise in this setup and the methods applied are of a more analytic nature than for closed surfaces. Moreover, we emphasise the existence of a unique limit point for any Teichmüller ray in  $T(\mathbb{D})$  which is not true for the Teichmüller spaces of closed surfaces.

The convergence of Teichmüller geodesic rays is in the weak\* topology while the convergence of earthquake paths is in the *uniform* weak\* topology. We prove in §5

**Proposition 2.** *There exists an integrable holomorphic quadratic differential  $\varphi$  on the unit disk  $\mathbb{D}$  such that the corresponding Teichmüller geodesic ray does not converge in the uniform weak\* topology.*

**Remark 4.** Thus while any Teichmüller geodesic ray in  $T(\mathbb{D})$  converges in the weak\* topology, there exist Teichmüller rays that do not converge in the strongest possible sense (the uniform weak\* topology) in  $T(\mathbb{D}) \cup PML_{bdd}(\mathbb{D})$  unlike earthquake paths. Also note that the uniform weak\* topology and the weak\* topology agree on the space of geodesic currents of a closed surface.

Denote by  $A(\mathbb{D})$  the space of all integrable holomorphic quadratic differentials on the unit disk  $\mathbb{D}$ . Let  $PA(\mathbb{D}) = (A(\mathbb{D}) - \{0\}) / \sim$ , where  $\varphi \sim \varphi_1$  if there exists  $c > 0$  with  $\varphi = c\varphi_1$ . By definition, we have  $\mu_{c\varphi} = \mu_\varphi$  for any  $c > 0$ . Therefore we obtained a map

$$\mathcal{M} : PA(\mathbb{D}) \rightarrow PML_{bdd}(\mathbb{D})$$

given by

$$\mathcal{M}([\varphi]) = [\mu_\varphi],$$

where  $[\varphi]$  and  $[\mu_\varphi]$  are the projective classes of  $\varphi$  and  $\mu_\varphi$ , respectively.

**Theorem 2.** *The map*

$$\mathcal{M} : PA(\mathbb{D}) \rightarrow PML_{bdd}(\mathbb{D}); \mathcal{M} : [\varphi] \mapsto [\mu_\varphi]$$

*is injective.*

**Remark 5.** By Theorem 1 and Theorem 2, two different Teichmüller geodesic rays in  $T(\mathbb{D})$  starting at the basepoint of  $T(\mathbb{D})$  converge to different points in Thurston's boundary. On the other hand, Masur [13] proved that two Teichmüller geodesic rays corresponding to two holomorphic quadratic differentials whose vertical foliations decompose a compact surface into finitely many cylinders of the same topological type but different relative heights converge to the same point in Thurston's boundary.

For  $\varphi \in A(\mathbb{D})$ , denote by  $\nu_\varphi$  the measured lamination whose support is the geodesic lamination  $v_\varphi$  homotopic to the vertical foliation of  $\varphi$  and whose transverse measure is induced by the transverse measure

to the vertical foliation induced by  $\varphi$ . We recover the integral of  $|\varphi|$  using  $\mu_\varphi$  and  $\nu_\varphi$ , namely

$$\|\varphi\|_{L^1} = \int_{\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}} \frac{d\nu_\varphi}{d\mu_\varphi} d\nu_\varphi.$$

**Theorem 3.** *The map*

$$A(\mathbb{D}) \rightarrow ML_{bdd}(\mathbb{D}) \times ML_{bdd}(\mathbb{D})$$

*defined by*

$$\varphi \mapsto (\nu_\varphi, \mu_\varphi)$$

*is injective.*

The paper is organized as follows. In §2 we define the universal Teichmüller space  $T(\mathbb{D})$ , the space of geodesic currents, the Liouville current and Thurston's boundary to  $T(\mathbb{D})$ . In §3 we define modulus of a family of curves and find a relationship between the modulus and relative distance. Finally we give asymptotic relationship between the modulus and the Liouville current which is fundamental to our work. In §4 we study the limits of Teichmüller geodesic rays and prove Theorem 1. In §5 we give a counter-example to uniform weak\* convergence of Teichmüller geodesic rays. In §6 we study the relationship between the integrable holomorphic quadratic differentials and two measured laminations homotopic to the vertical foliation.

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## 2. THURSTON'S BOUNDARY VIA GEODESIC CURRENTS

We identify the unit disk  $\mathbb{D}$  with the hyperbolic plane; the visual boundary to  $\mathbb{D}$  is the unit circle  $\mathbb{S}^1$ . A homeomorphism  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is said to be *quasisymmetric* if there exists  $M \geq 1$  such that

$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M$$

for all circular arcs  $I, J$  with a common boundary point and disjoint interiors such that  $|I| = |J|$ , where  $|I|$  is the length of  $I$ . A homeomorphism is quasisymmetric if and only if it extends to a quasiconformal map of the unit disk, see e.g. [10].

**Definition 2.1.** The universal Teichmüller space  $T(\mathbb{D})$  consists of all quasisymmetric maps  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  that fix  $-i, 1, i \in \mathbb{S}^1$ .

If  $g : \mathbb{D} \rightarrow \mathbb{D}$  is a quasiconformal map, denote by  $K(g)$  its quasiconformal constant. The Teichmüller metric on  $T(\mathbb{D})$  is given by  $d(h_1, h_2) = \inf_g \log K(g)$ , where  $g$  runs over all quasiconformal extensions of the quasisymmetric map  $h_1 \circ h_2^{-1}$ . The Teichmüller topology is induced by the Teichmüller metric.

Bonahon's approach [2] to Thurston's boundary of the Teichmüller space  $T(S)$  of a closed surface  $S$  is to embed  $T(S)$  into the space of geodesic currents on  $S$ . A *geodesic current* on  $S$  is a positive Borel measure on the space of geodesics  $\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$  of the universal covering  $\mathbb{D}$  of  $S$  that is invariant under the action of the covering group  $\pi_1(S)$ . Each point in the Teichmüller space  $T(S)$  is a quasisymmetric map

$$h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

that conjugates the covering Fuchsian group  $\pi_1(S)$  onto another Fuchsian group.

The *Liouville measure*  $\mathcal{L}$  on the space of geodesic of  $\mathbb{D}$  is given by

$$\mathcal{L}(A) = \int_A \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2}$$

for any Borel set  $A \subset \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$ . If  $A = [a, b] \times [c, d]$  is a *box of geodesics* then

$$\mathcal{L}([a, b] \times [c, d]) = \log \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

To each quasisymmetric map  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  that conjugates  $\pi_1(S)$  onto another Fuchsian group, we assign the pull back  $h^*(\mathcal{L})$  of the Liouville measure. This assignment is a homeomorphism of  $T(S)$  onto its image in the space of geodesic currents for  $S$  when equipped with the weak\* topology (cf. [2]). The set of the asymptotic rays to the image of  $T(S)$  is identified with the projective measured laminations on  $S$ -the Thurston's boundary of  $T(S)$  (cf. [2]).

The universal Teichmüller space  $T(\mathbb{D})$  maps into the space of geodesic currents by taking the pull backs by quasimetric maps of the Liouville measure. There is no invariance condition on the quasimetric maps or on the pull backs of the Liouville measure. A geodesic current  $\alpha$  is *bounded* if

$$\sup_{[a,b] \times [c,d]} \alpha([a,b] \times [c,d]) < \infty$$

where the supremum is over all boxes of geodesics  $[a,b] \times [c,d]$  with  $\mathcal{L}([a,b] \times [c,d]) = \log 2$ . The pull backs  $h^*(\mathcal{L})$  for  $h$  quasimetric are bounded geodesic currents.

The space of bounded geodesic currents is endowed with the family of Hölder norms parametrized with the Hölder exponents  $0 < \nu \leq 1$  (cf. [18]). The pull backs of the Liouville measure define a homeomorphism of  $T(\mathbb{D})$  onto its image in the bounded geodesic currents; the homeomorphism is differentiable with a bounded derivative (cf. [19]) and, in fact, Otal [15] proved that the embedding is real-analytic. The asymptotic rays to the image of  $T(\mathbb{D})$  are identified with the space of projective bounded measured laminations (cf. [18]). Thus Thurston's boundary of  $T(\mathbb{D})$  is the space  $PML_{bdd}(\mathbb{D})$  of all projective bounded measured laminations on  $\mathbb{D}$  (and an analogous statement holds for any hyperbolic Riemann surface). Alternatively, the space of geodesic currents can be endowed with the uniform weak\* topology (for definition cf. [14]) and Thurston's boundary for  $T(\mathbb{D})$  is again  $PML_{bdd}(\mathbb{D})$  (cf. [20]).

### 3. THE ASYMPTOTICS OF THE MODULUS

Let  $\Gamma$  be a family of rectifiable curves in  $\mathbb{C}$ . An *admissible metric*  $\rho$  for  $\Gamma$  is a non-negative Borel measurable function on  $\mathbb{D}$  such that the  $\rho$ -length of each  $\gamma \in \Gamma$  is at least one, namely

$$l_\rho(\gamma) = \int_\gamma \rho(z) |dz| \geq 1.$$

The *modulus*  $\text{mod}(\Gamma)$  of the family  $\Gamma$  is given by

$$\text{mod}(\Gamma) = \inf_\rho \int_{\mathbb{D}} \rho(z)^2 dx dy$$

where the infimum is over all admissible metrics  $\rho$ .

We will mostly be interested in estimating moduli of families of curves in a domain  $\Omega \subset \mathbb{C}$  connecting two subsets of the boundary of  $\Omega$ . Thus, given  $E, F \subset \partial\Omega$  we denote  $(E, F; \Omega)$  the family of rectifiable curves  $\gamma$  having one endpoint in  $E$  and the other endpoint in  $F$ . When  $\Omega$  is the unit disc  $\mathbb{D}$  and  $(a, b, c, d)$  is a quadruple of distinct points on the boundary circle  $\mathbb{S}^1$  given in the counterclockwise order we denote

$$\Gamma_{[a,b] \times [c,d]} = ((a, b), (c, d); \mathbb{D}).$$

Lemma 3.1 below, summarizes some of the main properties of the modulus, which we will use repeatedly throughout the paper. We refer the reader to [5, 10, 22] for the proofs of these properties below and for further background on modulus.

If  $\Gamma_1$  and  $\Gamma_2$  are curve families in  $\mathbb{C}$ , we will say that  $\Gamma_1$  *overflows*  $\Gamma_2$  and will write  $\Gamma_1 > \Gamma_2$  if every curve  $\gamma_1 \in \Gamma_1$  contains some curve  $\gamma_2 \in \Gamma_2$ .

**Lemma 3.1.** *Let  $\Gamma_1, \Gamma_2, \dots$  be curve families in  $\mathbb{C}$ . Then*

1. MONOTONICITY: *If  $\Gamma_1 \subset \Gamma_2$  then  $\text{mod}(\Gamma_1) \leq \text{mod}(\Gamma_2)$ .*
2. SUBADDITIVITY:  *$\text{mod}(\bigcup_{i=1}^\infty \Gamma_i) \leq \sum_{i=1}^\infty \text{mod}(\Gamma_i)$ .*
3. OVERFLOWING: *If  $\Gamma_1 < \Gamma_2$  then  $\text{mod}\Gamma_1 \geq \text{mod}\Gamma_2$ .*

Heuristically modulus of  $(E, F; \Omega)$  measures the amount of curves connecting  $E$  and  $F$  in the  $\Omega$ . The more “short” curves there are the bigger the modulus is. This heuristic may be made precise using a notion of relative distance  $\Delta(E, F)$ , which we define next.

Given two continua  $E$  and  $F$  in  $\mathbb{C}$  we denote

$$(2) \quad \Delta(E, F) := \frac{\text{dist}(E, F)}{\min\{\text{diam}E, \text{diam}F\}},$$

i.e.  $\Delta(E, F)$  is the *relative distance* between  $E$  and  $F$  in  $\mathbb{C}$ .

**Lemma 3.2.** *For every pair of continua  $E, F \subset \mathbb{C}$  we have*

$$(3) \quad \text{mod}(E, F; \mathbb{C}) \leq \pi \left( 1 + \frac{1}{2\Delta(E, F)} \right)^2.$$

*Proof.* Let  $\delta := \text{dist}(E, F)$  and  $\Gamma_E^\delta$  be the family of curves  $\gamma \subset \mathbb{C}$  such that  $\gamma(0) \in E$  and  $\text{dist}(\gamma(1), E) \geq \delta$ . Then  $(E, F; \mathbb{C}) \subset \Gamma_E^\delta$  and similarly  $(E, F; \mathbb{C}) \subset \Gamma_F^\delta$ . Therefore,

$$(4) \quad \text{mod}(E, F; \mathbb{C}) \leq \min\{\text{mod}\Gamma_E^\delta, \text{mod}\Gamma_F^\delta\}.$$

Denoting by  $E^\delta$  the  $\delta$ -neighborhood of the set  $E \subset \mathbb{C}$ , we note, that

$$\rho(z) = \delta^{-1} \chi_{E^\delta}(z)$$

is admissible for  $\Gamma_E^\delta$ . Therefore, we have

$$(5) \quad \begin{aligned} \text{mod}\Gamma_E^\delta &\leq \int_{E^\delta} (\delta^{-1})^2 dx dy = \delta^{-2} \mathcal{H}^2(E^\delta) \leq \delta^{-2} \pi \left( \frac{\text{diam}E + 2\delta}{2} \right)^2 \\ &= \pi \left( 1 + \frac{\text{diam}E}{2\text{dist}(E, F)} \right)^2, \end{aligned}$$

where  $\mathcal{H}^2(E^\delta)$  is the Euclidean area of  $E^\delta$ . Combining inequalities (4) and (5) we obtain (3).  $\square$

**Corollary 3.3.** *Let  $E_n$  and  $F_n$ ,  $n \in \mathbb{N}$ , be a sequence of pairs of continua in  $\mathbb{C}$ . If the sequence  $\Delta(E_n, F_n)$  is bounded away from 0 then  $\text{mod}(E_n, F_n; \mathbb{C})$  is bounded.*

**Remark 3.4.** The previous lemma is very weak for large  $\Delta(E, F)$ , since it is in fact easy to see that  $\text{mod}(E, F; \mathbb{C})$  tends to 0 as  $\Delta(E, F) \rightarrow \infty$ . But we will not need this estimate in the present paper and will refer the interested reader to Heinonen's book [8] for relations between the modulus and relative distance.

The following lemma is an easy consequence of the asymptotic properties of the moduli (cf. [10]).

**Lemma 3.5** (cf. [7]). *Let  $(a, b, c, d)$  be a quadruple of points on  $\mathbb{S}^1$  in the counterclockwise order. Let  $\Gamma_{[a,b] \times [c,d]}$  consist of all differentiable curves  $\gamma$  in  $\mathbb{D}$  which connect  $[a, b] \subset \mathbb{S}^1$  with  $[c, d] \subset \mathbb{S}^1$ . Then*

$$\text{mod}(\Gamma_{[a,b] \times [c,d]}) - \frac{1}{\pi} \mathcal{L}([a, b] \times [c, d]) - \frac{2}{\pi} \log 4 \rightarrow 0$$

as  $\text{mod}(\Gamma_{[a,b] \times [c,d]}) \rightarrow \infty$ , where  $\mathcal{L}$  is the Liouville measure.

**Remark 3.6.** Note that simultaneously  $\text{mod}(\Gamma_{[a,b] \times [c,d]}) \rightarrow \infty$  and  $\mathcal{L}([a, b] \times [c, d]) \rightarrow \infty$ .

#### 4. THE CONVERGENCE OF TEICHMÜLLER RAYS

Let  $\varphi$  be an integrable holomorphic quadratic differential on the unit disk  $\mathbb{D}$ . In other words,  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$  is holomorphic and

$$\|\varphi\|_{L^1} = \iint_{\mathbb{D}} |\varphi(z)| dx dy < \infty.$$

A point  $z \in \mathbb{D}$  is said to be *regular* for  $\varphi$  if  $\varphi(z) \neq 0$ . In a neighborhood of every regular point of  $\varphi$  the parameter  $w$  given by path integral  $w = \int \sqrt{\varphi(z)} dz$  is called a *natural parameter* for  $\varphi$ . The holomorphic quadratic differential  $\varphi(z) dz^2$  has representation  $dw^2$  in the natural parameter  $w$ . Moreover, if  $w'$  is another natural parameter then  $w' = \pm w + \text{const}$  at their intersection (cf. [21]).

A *vertical arc* for  $\varphi$  is a differentiable arc  $\gamma : (a, b) \rightarrow \mathbb{D}$  that passes only through regular points of  $\varphi$  and that satisfies  $\varphi(\gamma(t))\gamma'(t)^2 < 0$  for all  $t \in (a, b)$ . Equivalently, a vertical arc is an inverse image of a Euclidean vertical arc in the natural parameter  $w$ . A *vertical trajectory* of  $\varphi$  is a *maximal* vertical arc. Similarly, a *horizontal arc* for  $\varphi$  is a differentiable arc  $\gamma : (a, b) \rightarrow \mathbb{D}$  that passes through regular points and satisfies  $\varphi(\gamma(t))\gamma'(t)^2 > 0$  for all  $t \in (a, b)$  (cf. [21]).

Each end of a vertical trajectory either accumulates to a zero of  $\varphi(z)$  or to the boundary  $\mathbb{S}^1$  of  $\mathbb{D}$ . In particular, if an end of a vertical trajectory of  $\varphi$  accumulates to the boundary  $\mathbb{S}^1$  then the limit set on  $\mathbb{S}^1$  consists of a single point and we say that the vertical trajectory has an endpoint on  $\mathbb{S}^1$  (cf. [21]). The set of zeroes of  $\varphi$  is countable and therefore only countably many vertical trajectories have an endpoint at a zero of  $\varphi$ . Any vertical trajectory of  $\varphi$  not in the above countable set has two distinct endpoints on  $\mathbb{S}^1$  (cf. [21]).

We define the *width* of a curve  $\gamma$  in  $\mathbb{D}$ . By Strebel [21], the unit disk  $\mathbb{D}$  can be decomposed into countably many disjoint open strips  $S(\beta_i)$  up to a countable family of vertical trajectories, where  $\beta_i$  is an open horizontal arc and  $S(\beta_i)$  is the union of vertical trajectories intersecting  $\beta_i$ . The strips  $S(\beta_i)$  are open and simply connected. The natural parameter

$$w = \int \sqrt{\varphi(z)} dz$$

is well-defined on each  $S(\beta_i)$  since  $S(\beta_i)$  is simply connected and does not contain any zeroes of  $\varphi$ . Any Borel  $A \subset \beta_i$  has well-defined width

$$\text{width}(A) = \int_A |\sqrt{\varphi(z)} dz|.$$

If  $\gamma \subset S(\beta_i)$ , denote by  $\pi_{\beta_i}(\gamma)$  the projection of  $\gamma$  onto  $\beta_i$  along the vertical trajectories. Then the width of  $\gamma$  is defined by

$$\text{width}(\gamma) = \int_{\pi_{\beta_i}(\gamma)} |\sqrt{\varphi(z)} dz|.$$

Assume that  $\gamma$  is not contained in a single strip. Consider the collection of Borel sets  $\pi_{\beta_i}(\gamma \cap S(\beta_i))$  for all  $i$  with  $\gamma \cap S(\beta_i) \neq \emptyset$ . We define the *width* of  $\gamma$  by

$$\text{width}(\gamma) = \sum_{i=1}^{\infty} \text{width}(\gamma \cap S(\beta_i)).$$

The definition  $\text{width}(\gamma)$  is given in terms of the strips  $S(\beta_i)$ . To see that  $\text{width}(\gamma)$  is independent of the choice of the strips, let  $S(\beta'_j)$  be another countable collection of disjoint open strips that covers  $\mathbb{D}$  up to countable union of singular vertical trajectories. The two strips  $S(\beta_i)$  and  $S(\beta'_j)$  are either disjoint or they intersect in an open strip  $S(\beta_{i,j})$ , where  $\beta_{i,j}$  is an open subinterval on  $\beta_i$  which can be homotoped modulo vertical trajectories to subinterval of  $\beta'_j$ . The homotopy is measure preserving for  $\int_* |\sqrt{\varphi(z)} dz|$ . Since  $\beta_i - \cup_j \beta_{i,j}$  is at most countable (which is of measure zero), it follows that

$$\text{width}(\gamma \cap S(\beta_i)) = \sum_j \text{width}(\gamma \cap S(\beta_{i,j})).$$

This implies that  $\text{width}(\gamma)$  is independent of the choice of the covering by the strips.

**Proposition 4.1.** *Let  $\Gamma = \Gamma([a, b] \times [c, d])$  be the family of rectifiable arcs in  $\mathbb{D}$  with one endpoint in  $[a, b] \subset \mathbb{S}^1$  and the other endpoint in  $[c, d] \subset \mathbb{S}^1$ . Denote by  $T_\epsilon$  the Teichmüller map of  $\mathbb{D}$  that shrinks the vertical trajectories of  $\varphi$  by the multiplicative constant  $\epsilon > 0$ . Then*

$$\limsup_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma)) \leq \text{mod}(\Gamma_v([a, b], [c, d]))$$

where  $\Gamma_v([a, b], [c, d])$  is the set of vertical trajectories of  $\varphi$  with one endpoint in  $[a, b]$  and the other endpoint in  $[c, d]$ .

*Proof.* By Strebel [21], almost every point of  $\mathbb{S}^1$  is at a finite distance from an interior point of  $\mathbb{D}$  in the path metric  $\int_* \sqrt{|\varphi(z)} dz|$ , called  $\varphi$ -metric. Let  $a', b', c', d' \in \mathbb{S}^1$  be on finite distances from an interior point such that  $[a, b] \subset [a', b']$  and  $[c, d] \subset [c', d']$ . Let  $\Gamma' = \Gamma([a', b'] \times [c', d'])$ .

Namely, let (cf. Figure 2)

$$\Gamma' = \{\gamma \mid \gamma \text{ is rectifiable and has endpoints in } [a', b'] \text{ and } [c', d']\}.$$

Since  $\Gamma \subset \Gamma'$ , we have  $\text{mod}(T_\epsilon(\Gamma)) \leq \text{mod}(T_\epsilon(\Gamma'))$ . Let  $l_{a', b'}$  and  $l_{c', d'}$  be two simple non-intersecting differentiable arcs in  $\mathbb{D}$  with endpoints  $a', b'$  and  $c', d'$ , respectively. Let  $\mathbb{D}'$  be the subset of  $\mathbb{D}$  with boundary consisting of arcs  $l_{a', b'}$ ,  $[b', c'] \subset \mathbb{S}^1$ ,  $l_{c', d'}$  and  $[d', a'] \subset \mathbb{S}^1$ . Let  $\Gamma'' = \Gamma(l_{a', b'} \times l_{c', d'})$  be the family of rectifiable curves in  $\mathbb{D}'$  that connect  $l_{a', b'}$  and  $l_{c', d'}$ . Then the family  $\Gamma'$  overflows the family  $\Gamma''$  and we have

$$(6) \quad \text{mod}(T_\epsilon(\Gamma')) \leq \text{mod}(T_\epsilon(\Gamma'')).$$

Fix  $\eta > 0$  and define

$$\Gamma''_{>\eta} = \{\gamma \in \Gamma'' \mid \text{width}(\gamma) > \eta\}$$

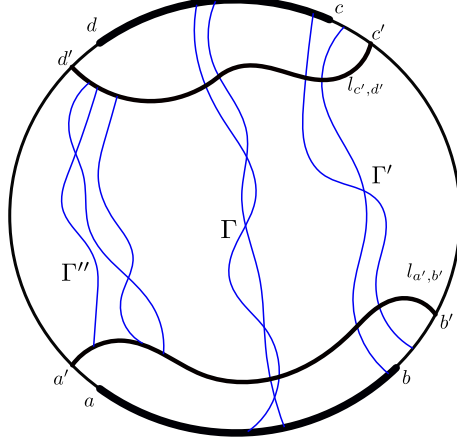


FIGURE 2. The curve families  $\Gamma$ ,  $\Gamma'$  and  $\Gamma''$ .

and

$$\Gamma''_{\leq \eta} = \{\gamma \in \Gamma'' \mid \text{width}(\gamma) \leq \eta\}.$$

By the subadditivity of the modulus

$$\text{mod}(T_\epsilon(\Gamma'')) \leq \text{mod}(T_\epsilon(\Gamma''_{>\eta})) + \text{mod}(T_\epsilon(\Gamma''_{\leq \eta})).$$

First consider  $\text{mod}(T_\epsilon(\Gamma''_{>\eta}))$ . Define the metric  $\rho_\epsilon(w) = \frac{1}{\eta} |\sqrt{\varphi_\epsilon(w)} dw^2|$  for  $w \in \mathbb{D}'_\epsilon$ , where  $\varphi_\epsilon$  is the terminal holomorphic quadratic differential on  $T_\epsilon(\mathbb{D}') = \mathbb{D}'_\epsilon$  (cf. [4]). Recall that the terminal quadratic differential on  $T_\epsilon(\mathbb{D}')$  is obtained as follows. Let  $\zeta$  be the natural parameter of  $\varphi$  on  $\mathbb{D}'$ , i.e.  $d\zeta^2 = \varphi(z) dz^2$ ; let  $\omega = T_{\epsilon, \zeta}(\zeta)$ , where  $T_{\epsilon, \zeta}$  shrinks the vertical direction of  $\zeta$  by the multiplicative constant  $\epsilon > 0$ . Then the terminal quadratic differential  $\varphi_\epsilon$  is defined in the image of the natural parameter as  $\varphi_\epsilon(\omega) d\omega^2 = d\omega^2$ . If  $w = T_\epsilon(z)$  then  $\varphi_\epsilon(w) dw^2 = d\omega^2$ .

The metric  $\rho_\epsilon$  is admissible for  $T_\epsilon(\Gamma''_{>\eta})$  since  $\text{width}(T_\epsilon(\gamma)) > \eta$  for all  $\epsilon > 0$  and all  $\gamma \in T_\epsilon(\Gamma''_{>\eta})$ . Then

$$\text{mod}(T_\epsilon(\Gamma''_{>\eta})) \leq \iint_{T_\epsilon(\mathbb{D}')} \rho_\epsilon(w)^2 dA = \frac{\epsilon}{\eta^2} \iint_{\mathbb{D}'} |\varphi(w)| dA$$

which gives

$$(7) \quad \limsup_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma''_{>\eta})) = 0,$$

since  $\phi$  is integrable.

We estimate  $\text{mod}(T_\epsilon(\Gamma''_{\leq \eta}))$ . Let  $z_0 \in \mathbb{D}'$  be fixed. Denote by  $d^\varphi$  the path metric defined by integrating  $|\sqrt{\varphi(z)} dz^2|$  i.e. the  $\varphi$ -metric. Let  $d_0 = \max_{z \in l_{a',b'} \cup l_{c',d'}} d^\varphi(z_0, z)$ . For  $R > 0$  define  $\mathbb{D}'_R = \{z \in \mathbb{D}' \mid d^\varphi(z_0, z) \leq R\}$ . Given  $\epsilon_1 > 0$  there exists  $R > 2d_0$  such that

$$\iint_{\mathbb{D}' - \mathbb{D}'_R} |\varphi(z)| dA < \epsilon_1.$$

Denote by  $\Gamma_v(l_{a',b'}, l_{c',d'})$  the set of vertical trajectories  $\gamma$  connecting  $l_{a',b'}$  with  $l_{c',d'}$ . The choice  $R > 2d_0$  and the fact that the vertical trajectories are geodesics for  $d^\varphi$  implies that  $\Gamma_v(l_{a',b'}, l_{c',d'}) \subset \mathbb{D}'_R$ . From now on we choose  $R = R(\epsilon_1)$  as above.

For  $M > 0$ , define  $(\Gamma''_{\leq \eta})_M = \{\gamma \in \Gamma''_{\leq \eta} \mid \gamma \subset \mathbb{D}'_M\}$ . Note that

$$\Gamma''_{\leq \eta} = (\Gamma''_{\leq \eta})_{R+1} \cup [\Gamma''_{\leq \eta} \setminus (\Gamma''_{\leq \eta})_{R+1}]$$

which gives

$$\text{mod}(T_\epsilon(\Gamma''_{\leq \eta})) \leq \text{mod}(T_\epsilon((\Gamma''_{\leq \eta})_{R+1})) + \text{mod}(T_\epsilon(\Gamma''_{\leq \eta} \setminus (\Gamma''_{\leq \eta})_{R+1})).$$

Since  $T_\epsilon$  is  $\epsilon^{-1}$ -quasiconformal, we have

$$\epsilon \cdot \text{mod}(T_\epsilon(\Gamma''_{\leq \eta} \setminus (\Gamma''_{\leq \eta})_{R+1})) \leq \epsilon \cdot \epsilon^{-1} \cdot \text{mod}(\Gamma''_{\leq \eta} \setminus (\Gamma''_{\leq \eta})_{R+1}) = \text{mod}(\Gamma''_{\leq \eta} \setminus (\Gamma''_{\leq \eta})_{R+1}).$$



Define metric  $\rho(z) = \sqrt{|\varphi(z)dz^2|}$  for  $z \in \mathbb{D}' - \mathbb{D}'_R$  and  $\rho(z) = 0$  otherwise. Then  $\rho(z)$  is admissible for the family  $\Gamma''_{\leq \eta} \setminus (\Gamma''_{\leq \eta})_{R+1}$ . Thus

$$(8) \quad \limsup_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma''_{\leq \eta} - (\Gamma''_{\leq \eta})_{R+1})) \leq \iint_{\mathbb{D}' - \mathbb{D}'_R} |\varphi(z)| dA < \epsilon_1.$$

We estimate  $\text{mod}(T_\epsilon((\Gamma''_{\leq \eta})_{R+1}))$ . Note that  $\mathbb{D}'_{R+1}$  is a compact metric space for the distance  $d^\varphi$ . Similar to the above

$$\epsilon \cdot \text{mod}(T_\epsilon((\Gamma''_{\leq \eta})_{R+1})) \leq \text{mod}((\Gamma''_{\leq \eta})_{R+1}).$$

By Keith [9], we have that

$$(9) \quad \limsup_{\eta \rightarrow 0^+} \text{mod}((\Gamma''_{\leq \eta})_{R+1}) \leq \text{mod}(\limsup_{\eta \rightarrow 0^+} (\Gamma''_{\leq \eta})_{R+1}).$$

Recall that a sequence  $\{\Gamma_n\}$  of families of curves converges to a family  $\Gamma$  if for each  $\gamma \in \Gamma$  there exists a subsequence  $\gamma_{n_k} \in \Gamma_{n_k}$  such that uniformly Lipschitz parameterizations of  $\gamma_{n_k}$  converge to  $\gamma$  as functions when  $n_k \rightarrow \infty$ , and if the limit of each convergent subsequence is a curve in  $\Gamma$  (cf. [9]).

We establish that

$$(10) \quad \limsup_{\eta \rightarrow 0^+} (\Gamma''_{\leq \eta})_{R+1} = \Gamma_v(l_{a',b'}, l_{c',d'}).$$

Let  $\gamma_n : I \rightarrow \mathbb{D}'_{R+1}$  be a sequence of uniformly Lipschitz parametrizations of curves in  $(\Gamma''_{\leq \eta_n})_{R+1}$  with  $\eta_n \rightarrow 0$  as  $n \rightarrow \infty$  that converges to  $\gamma : I \rightarrow \mathbb{D}'_{R+1}$ . Then

$$\text{width}(\gamma) = 0.$$

Indeed,  $\text{width}(\gamma) = c > 0$  implies that  $\text{width}(\gamma_n) > c/2 > 0$  for all  $n$  large enough. This contradicts  $\gamma_n \in (\Gamma''_{\leq \eta_n})_{R+1}$ .

Since  $\text{width}(\gamma) = 0$ , this implies  $\gamma \in \Gamma_v(l_{a',b'}, l_{c',d'})$  or that  $\gamma$  is composed of several vertical trajectories that meet at a zero of  $\varphi$ . The later curves are at most countable and their modulus is zero, so we can ignore them. Since  $\Gamma_v(l_{a',b'}, l_{c',d'}) \subset (\Gamma''_{\leq \eta})_{R+1}$  by our choice of  $R > 0$ , we obtain (10). Then (6), (7), (8) and (10) imply that

$$(11) \quad \limsup_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma)) \leq \text{mod}(\Gamma_v(l_{a',b'}, l_{c',d'})).$$

We prove that  $\Gamma_v(l_{a',b'}, l_{c',d'})$  can be replaced by  $\Gamma_v([a, b] \times [c, d])$  in (11). Note that (11) is true for all  $l_{a',b'}$  and  $l_{c',d'}$ . Choose a sequence  $l_{a',b'}^k$  and  $l_{c',d'}^k$  such that  $l_{a',b'}^k \rightarrow [a', b'] \subset \mathbb{S}^1$  and  $l_{c',d'}^k \rightarrow [c', d'] \subset \mathbb{S}^1$  as  $k \rightarrow \infty$  in the Hausdorff topology on closed subsets of  $\bar{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$ . Denote by  $\mathbb{D}'_k$  the subset of  $\mathbb{D}$  corresponding to  $l_{a',b'}^k$  and  $l_{c',d'}^k$ . Define

$$\Gamma_v^k([a', b'], [c', d']) := \Gamma_v([a', b'], [c', d']) \cap \mathbb{D}'_k.$$

We claim that

$$(12) \quad \lim_{k \rightarrow \infty} \text{mod}(\Gamma_v(l_{a',b'}^k, l_{c',d'}^k) - \Gamma_v^k([a', b'], [c', d'])) = 0.$$

Indeed, let  $C > 0$  be the lower bound on the distance  $d^\varphi$  between  $l_{a',b'}^k$  and  $l_{c',d'}^k$  over all  $k$ . Then  $\rho(z) = \frac{1}{C} \sqrt{|\varphi(z)|} |dz|$  is admissible for  $\Gamma_v(l_{a',b'}^k, l_{c',d'}^k)$ . Let  $A_k$  be the union of the (complete) vertical trajectories in  $\mathbb{D}$  that connect  $l_{a',b'}^k$  and  $l_{c',d'}^k$  and do not connect  $[a', b']$  and  $[c', d']$ . Then  $A_k \supset A_{k+1}$  for all  $k$  (since we can choose  $l_{a',b'}^k$  and  $l_{c',d'}^k$  such that  $\mathbb{D}'_k \subset \mathbb{D}'_{k+1}$ ).

We claim that  $\bigcap_{k=1}^\infty A_k = \emptyset$ . Assume that a horizontal trajectory  $\gamma$  belongs to the union that makes  $A_k$ . Then there exists either a Euclidean neighborhood of  $[a', b']$  or a Euclidean neighborhood of  $[c', d']$  in  $\mathbb{D} = \mathbb{D} \cup \mathbb{S}^1$  such that  $\gamma$  is disjoint from this neighborhood. There exists  $k' > k$  such that  $\gamma$  does not intersect either  $l_{a',b'}^{k'}$  or  $l_{c',d'}^{k'}$ . Thus  $\gamma$  does not belong to  $\bigcap_{k=1}^\infty A_k$  and  $\bigcap_{k=1}^\infty A_k = \emptyset$ . This gives

$$\iint_{A_k} |\varphi(z)| dx dy \rightarrow 0$$

as  $k \rightarrow \infty$  and (12) follows. From (11) and (12) we get

$$(13) \quad \limsup_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma)) \leq \lim_{k \rightarrow \infty} \text{mod}(\Gamma_v^k([a', b'], [c', d'])).$$

By Keith [9], we have that

$$\lim_{k \rightarrow \infty} \text{mod}(\Gamma_v^k([a', b'], [c', d'])) \leq \text{mod}(\limsup_{k \rightarrow \infty} \Gamma_v^k([a', b'], [c', d']))$$

where  $\limsup_{k \rightarrow \infty} \Gamma_v^k([a', b'], [c', d'])$  is for the Euclidean metric on  $\bar{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$ . For every point of  $\gamma$  which is not a zero of  $\varphi$ , there exists an open subarc of  $\gamma$  containing the point that is a part of a vertical trajectory of  $\varphi$  because  $\gamma_k$  are vertical trajectories. Moreover the limit  $\gamma$  has one endpoint in  $[a', b']$  and the other endpoint in  $[c', d']$  because  $\gamma_k$  has one endpoint on  $l_{a', b'}^k$  and one endpoint on  $l_{c', d'}^k$ , and  $l_{a', b'}^k$  converges to  $[a', b']$  and  $l_{c', d'}^k$  converges to  $[c', d']$ . Therefore every limit  $\gamma$  is a vertical trajectory that necessarily belongs to  $\Gamma_v([a', b'], [c', d'])$  or it is composed of several vertical trajectories meeting at zeros of  $\varphi$ . The later family is countable and of zero modulus and without loss of generality we ignore it. Therefore

$$(14) \quad \limsup_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma)) \leq \text{mod}(\Gamma_v([a', b'], [c', d'])).$$

We choose sequences  $[a'_k, b'_k] \supset [a, b]$  and  $[c'_k, d'_k] \supset [c, d]$  on finite distance from  $z_0$  such that  $a'_k \rightarrow a$ ,  $b'_k \rightarrow b$ ,  $c'_k \rightarrow c$  and  $d'_k \rightarrow d$  as  $k \rightarrow \infty$ . The inequality (14) holds for these sequences and we need to prove that it holds for  $\Gamma_v([a, b], [c, d])$  as well. It is enough to prove that

$$\text{mod}(\Gamma_v([a'_k, b'_k], [c'_k, d'_k]) - \Gamma_v([a, b], [c, d])) \rightarrow 0.$$

Let  $\mathbb{D}_k$  be the union of vertical trajectories in  $\Gamma_v([a'_k, b'_k], [c'_k, d'_k]) - \Gamma_v([a, b], [c, d])$  and note  $\cap_{k=1}^\infty \mathbb{D}_k = \emptyset$ . Define  $\rho(z) = 1/l_v(z)\sqrt{|\varphi(z)dz^2|}$  for  $z \in \mathbb{D}_k$  and  $\rho(z) = 0$  otherwise, where  $l_v(z)$  is the length of the vertical trajectory through  $z$  with respect to the metric  $d^\varphi$ . Then  $\rho$  is allowable metric for the family  $\Gamma_v([a'_k, b'_k], [c'_k, d'_k]) - \Gamma_v([a, b], [c, d])$ . We have

$$\text{mod}(\Gamma_v([a'_k, b'_k], [c'_k, d'_k]) - \Gamma_v([a, b], [c, d])) \leq \iint_{\mathbb{D}_k} \frac{1}{l_v(z)^2} |\varphi(z)| dx dy.$$

We claim that  $l_v(z)$  has a positive lower bound in  $\mathbb{D}_k$ . Indeed, since intervals  $[a'_k, b'_k]$  and  $[c'_k, d'_k]$  are disjoint and decreasing, their distance in  $d^\varphi$  metric is positive which implies that any vertical trajectories connecting them must have lengths bounded below by a positive constant. Thus  $\frac{1}{l_v(z)^2}$  is bounded above. Then  $\cap_{k=1}^\infty \mathbb{D}_k = \emptyset$  implies that  $\iint_{\mathbb{D}_k} \frac{1}{l_v(z)^2} |\varphi(z)| dx dy \rightarrow 0$  as  $k \rightarrow \infty$ . The proof is finished.  $\square$

**Theorem 4.2.** *Let  $\Gamma$  be the family of rectifiable arcs in  $\mathbb{D}$  with one endpoint in  $[a, b] \subset \mathbb{S}^1$  and the other endpoint in  $[c, d] \subset \mathbb{S}^1$ . Denote by  $T_\epsilon$  the Teichmüller map of  $\mathbb{D}$  that shrinks the vertical trajectories of  $\varphi$  by the multiplicative constant  $\epsilon > 0$ . Then*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma)) = \text{mod}(\Gamma_v([a, b], [c, d]))$$

where  $\Gamma_v([a, b], [c, d])$  is the set of vertical trajectories with one endpoint in  $[a, b]$  and the other endpoint in  $[c, d]$ .

*Proof.* We keep the notation as in the proof of Proposition 4.1. Since  $\Gamma_v([a, b], [c, d]) \subset \Gamma$ , it follows that  $\text{mod}(\Gamma_v([a, b], [c, d])) \leq \text{mod}(\Gamma)$ . Because  $\Gamma_v([a, b], [c, d])$  consists of only vertical trajectories, it follows that

$$\epsilon \cdot \text{mod}(T_\epsilon(\Gamma_v([a, b], [c, d]))) = \text{mod}(\Gamma_v([a, b], [c, d])).$$

Thus

$$\text{mod}(\Gamma_v([a, b], [c, d])) \leq \liminf_{\epsilon \rightarrow 0^+} \epsilon \cdot \text{mod}(T_\epsilon(\Gamma)).$$

The opposite inequality is obtained in Proposition 4.1 and theorem follows.  $\square$

We give an equivalent definition of  $\text{mod}(\Gamma_v([a, b], [c, d]))$ .

**Proposition 4.3.** *Let  $\varphi$  be an integrable holomorphic quadratic differential on the unit disk  $\mathbb{D}$ . Then*

$$\text{mod}(\Gamma_v([a, b], [c, d])) = \int_I \frac{1}{l(z)} |\text{Re}(\sqrt{\varphi(z)} dz)|$$

where  $I$  is at most countable set of horizontal arcs that intersects each trajectory of  $\Gamma_v([a, b], [c, d])$  in one point and no other vertical trajectories up to countably many of them, and  $l(z)$  is the length of the vertical trajectory through  $z$ .

*Proof.* The metric  $\rho(z) = \frac{1}{l(z)}|\sqrt{\varphi(z)}dz|$  is allowable for the family  $\Gamma_v([a, b], [c, d])$  and thus  $\text{mod}(\Gamma_v([a, b], [c, d])) \leq \int_I \frac{1}{l(z)}|\sqrt{\varphi(z)}dz|$ .

We claim that  $\rho(z)$  is extremal metric for the family  $\Gamma_v([a, b], [c, d])$  which proves that we have equality above. Using Beurling's criterion of sufficiency for extremal metrics [1], we need to show that if  $\int_\gamma h_0(z)|dz| \geq 0$  for all  $\gamma \in \Gamma_v([a, b], [c, d])$  and some  $h_0 : \mathbb{D} \rightarrow \mathbb{R}$  then we have  $\iint_{\mathbb{D}} h_0(z)\rho(z)^2 dx dy \geq 0$ . By transferring the integration to the natural parameter, we get that  $\gamma$  are subsets of vertical lines which implies  $|dz| = dy$  and  $\rho(z) = 1/l(z)$ . Note that  $l(z) = l(x)$  is independent of  $y$ . Then  $\int_\gamma h_0(z)|dz| = \int_I h_0(z)dy \geq 0$  and multiplying with  $1/l(x)^2$  and an integration in the  $x$  direction gives the desired inequality (cf. [7]).  $\square$

Define a measured lamination  $\mu_\varphi$  as follows. The support of  $\mu_\varphi$  is a geodesic lamination  $v_\varphi$  obtained by taking geodesics in  $\mathbb{D}$  which are homotopic to the vertical trajectories of  $\varphi$  relative their endpoints on  $\mathbb{S}^1$ , i.e. a geodesic in the support  $v_\varphi$  of  $\mu_\varphi$  has endpoints equal to a vertical trajectory of  $\varphi$ . For a box of geodesics  $[a, b] \times [c, d]$ , define

$$\mu_\varphi([a, b] \times [c, d]) = \text{mod}(\Gamma_v([a, b], [c, d])).$$

Note that  $\mu_\varphi$  is a measure on the space of geodesics (i.e. it is countable additive) by the above integration formula for  $\text{mod}(\Gamma_v([a, b], [c, d]))$ . Also note that  $\text{mod}(\Gamma([a, b], [c, d]))$  is not countably additive (since moduli are only countably subadditive) and hence it does not define a measure on  $\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$ .

**Proposition 4.4.** *Let  $\mu_\varphi$  be the measured geodesic lamination corresponding to an integrable holomorphic quadratic differential  $\varphi$  on  $\mathbb{D}$  as above. Then*

$$\mu_\varphi(\{a\} \times [c, d]) = 0$$

for all  $a \in \mathbb{S}^1$  and  $[c, d] \subset \mathbb{S}^1$  with  $a \notin [c, d]$ .

*Proof.* We recall that  $\mathbb{D}$  is covered by countably many mutually disjoint open strips  $S(\beta_i)$  up to countably many vertical trajectories. Assume on the contrary that  $\mu_\varphi(\{a\} \times [c, d]) > 0$ . Then there exists an open strip  $S(\beta_{i_0})$  such that

$$\int_{X_{i_0}} |\sqrt{\varphi(z)}dz^2| > 0,$$

where  $\beta_{i_0}$  is the open arc on a horizontal trajectory and  $X_{i_0} = \beta_{i_0} \cap \Gamma_v(\{a\} \times [c, d])$ . By the definition,  $\int_{X_{i_0}} |\sqrt{\varphi(z)}dz^2|$  is the horizontal measure in the natural parameter of  $\varphi$  of the vertical trajectories of  $\varphi$  intersecting  $\beta_{i_0}$ .

For  $z \in X_{i_0}$ , let  $l(z)$  be the length of the vertical trajectory through  $z$ . Since  $\varphi$  is integrable, we have that

$$\int_{X_{i_0}} l(z)|\sqrt{\varphi(z)}dz^2| < \infty$$

which implies that  $l(z) < \infty$  for a.e.  $z \in X_{i_0}$ .

Let  $z_1, z_2 \in X_{i_0}$  be such that there exists  $z'_1, z'_2 \in X_{i_0}$  with  $z_1 < z'_1 < z'_2 < z_2$  for a linear order on  $\beta_{i_0}$ , and  $l(z'_1)$  and  $l(z'_2)$  finite. Let  $\gamma_{z_i}, \gamma_{z'_i}$  be the maximal vertical rays starting at  $z_i, z'_i$  respectively that have  $a$  as their common endpoint. Note that vertical rays  $\gamma_{z_1}$  and  $\gamma_{z_2}$  do not intersect  $\beta_{i_0}$  except at their initial points because any two points in  $\mathbb{D}$  can be joined by at most one geodesic arc in the metric  $|\sqrt{\varphi(z)}dz^2|$  (cf. [21, Theorem 14.2.1, page 72]). Let  $[z_1, z_2]$  be the subarc of the vertical trajectory between  $z_1$  and  $z_2$ . Then  $\gamma_{z_1} \cup \gamma_{z_2} \cup [z_1, z_2] \cup \{a\}$  is the boundary of a simply connected domain  $U$  inside  $\mathbb{D}$ .

We claim that  $U$  is a Jordan domain. Indeed, since  $\gamma_{z_1}, \gamma_{z_2}$  and  $[z_1, z_2]$  are simple geodesic arcs which meet only at their endpoints, it follows that  $\gamma_{z_1} \cup \gamma_{z_2} \cup [z_1, z_2]$  is a Jordan arc. We parametrize it by a homeomorphism  $f : \mathbb{S}^1 - \{1\} \rightarrow \gamma_{z_1} \cup \gamma_{z_2} \cup [z_1, z_2]$  and extend  $f(1) = a$ . Then  $f$  is a bijection of  $\mathbb{S}^1$  and  $\partial U = \gamma_{z_1} \cup \gamma_{z_2} \cup [z_1, z_2] \cup \{a\}$ . Moreover,  $f$  is continuous at 1 since  $\gamma_{z_1}$  and  $\gamma_{z_2}$  accumulate to  $a$  and therefore  $\partial U$  is a Jordan curve.

For  $z \in [z_1, z_2]$ , let  $\gamma_z$  be the ray of the vertical trajectory with the initial point  $z$  that starts in the direction of  $U$ . Then  $\gamma_z$  never leaves  $U$  because it cannot intersect its boundary except at  $z$ . Moreover, the ray  $\gamma_z$  cannot contain critical points of  $\varphi$ . Indeed, if it does contain a critical point then there exist two vertical rays starting at the critical point which make a geodesic and whose both accumulation points on  $\mathbb{S}^1$  are equal to  $a$ . However, a geodesic must have two different accumulation points (cf. [21, Theorem 19.4 and

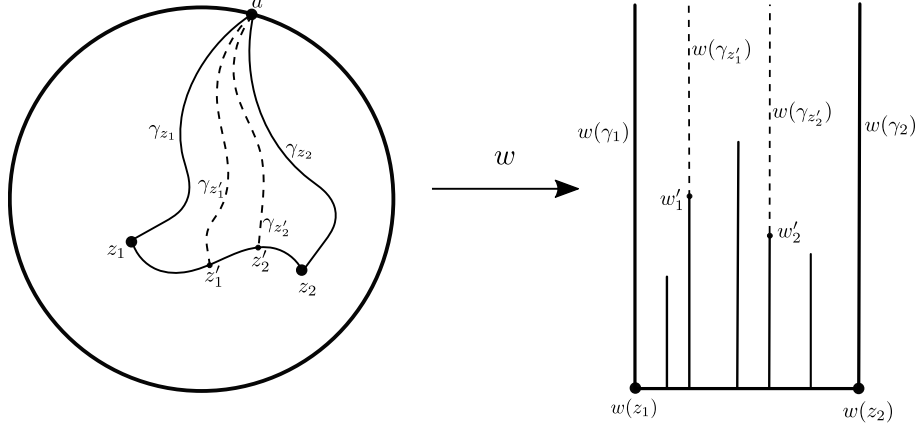


FIGURE 3. The prime ends of vertical trajectories.

Theorem 19.6]) which gives a contradiction. Therefore every vertical trajectory in  $U$  is non-critical and its full extension accumulates at  $a \in \mathbb{S}^1$  and intersects  $[z_1, z_2]$  in exactly one point. Therefore,  $U$  is foliated by  $\gamma_z$  for  $z \in (z_1, z_2)$ .

Consider the conformal mapping from  $U$  into  $\mathbb{C}$  using the natural parameter  $dw^2 = \varphi(z)dz^2$ . Since  $U$  is simply connected and without zeroes, the natural parameter is conformal on  $U$ . Caratheodory's theorem (cf. [16]) gives that  $w$  homeomorphically maps the boundary  $\partial U$  of  $U$  onto the prime ends of  $w(U)$ .

Since  $\gamma_{z'_1}$  and  $\gamma_{z'_2}$  have finite lengths, it follows that the endpoints  $w'_1$  and  $w'_2$  of vertical lines  $w(\gamma_{z'_1})$  and  $w(\gamma_{z'_2})$  are different in  $\partial w(U)$ . The arcs  $\gamma_{z'_1}$  and  $\gamma_{z'_2}$  define degenerate prime ends, namely prime ends whose imprints are  $w'_1$  and  $w'_2$ . Therefore the prime ends are different since  $w'_1 \neq w'_2$  (cf. Figure 3).

This is impossible since  $w$  maps  $a$  onto both prime ends. Contradiction. Thus we obtained that  $\mu_\varphi(\{a\} \times [c, d]) = 0$ .  $\square$

Putting the above statements together and using the fact that the asymptotics of the Liouville currents can be replaced by the asymptotics of the moduli of curves (cf. Lemma 3.5) gives

**Theorem 4.5.** *Let  $\varphi$  be an integrable holomorphic quadratic differential on  $\mathbb{D}$  and let  $T_\epsilon$  be the Teichmüller mapping that shrinks the vertical trajectories of  $\varphi$  by a multiplicative constant  $\epsilon > 0$ . The Teichmüller ray  $\epsilon \mapsto T_\epsilon$  for  $\epsilon > 0$  has a unique limit point  $[\mu_\varphi]$  on Thurston's boundary  $PML_{bdd}(\mathbb{D})$  of  $T(\mathbb{D})$  as  $\epsilon \rightarrow 0^+$ , where  $[\mu_\varphi]$  is the projective class of a bounded measured lamination  $\mu_\varphi$  corresponding to  $\varphi$ .*

*Proof.* The convergence  $T_\epsilon \rightarrow [\mu_\varphi]$  as  $\epsilon \rightarrow 0^+$  in the weak\* topology on measures follows immediately from Theorem 4.2, Proposition 4.4 and Lemma 3.5. It remains to be proved that  $\mu_\varphi$  is Thurston bounded.

Note that by the definition the measured lamination  $\mu_\varphi$  is independent under multiplication of  $\varphi$  by positive constants. Let  $[a, b] \times [c, d]$  be such that its Liouville measure satisfies

$$\mathcal{L}([a, b] \times [c, d]) = \log 2.$$

Denote by  $\Gamma([a, b], [c, d])$  the family of all rectifiable arcs in  $\mathbb{D}$  that have one endpoint in  $[a, b]$  and other endpoint in  $[c, d]$ . Then

$$\text{mod}(\Gamma([a, b], [c, d])) \leq \text{const}$$

for all  $\mathcal{L}([a, b] \times [c, d]) = \log 2$ . Since  $\Gamma_v([a, b], [c, d]) \subset \Gamma([a, b], [c, d])$ , we have that

$$\mu_\varphi([a, b] \times [c, d]) = \text{mod}(\Gamma_v([a, b], [c, d])) \leq \text{const}$$

and  $\|\mu_\varphi\|_{Th} < \infty$ .  $\square$

## 5. A COUNTER-EXAMPLE TO UNIFORM WEAK\* CONVERGENCE

Let  $\{\alpha_n\}_n$  and  $\alpha$  be geodesic currents on the space of geodesics  $G(\mathbb{H}) = \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$  of the hyperbolic plane  $\mathbb{H}$ . Namely,  $\{\alpha_n\}_n$  and  $\alpha$  are positive Radon measures on  $\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}$ . We say that  $\alpha_n$  converges

to  $\alpha$  in the *uniform* weak\* topology (cf. [18]) if for every continuous  $f : \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag} \rightarrow \mathbb{R}$  with support in the standard box  $[1, i] \times [-1, -i]$  we have

$$\sup_{[a,b] \times [c,d]} \left| \int_{[1,i] \times [-1,-i]} f[d\gamma_{[a,b] \times [c,d]}^*(\alpha_n - \alpha)] \right| \rightarrow 0$$

as  $n \rightarrow \infty$ , where the supremum is over all boxes  $[a, b] \times [c, d]$  of Liouville measure  $\log 2$  and  $\gamma_{[a,b] \times [c,d]}$  is the Möbius map taking the standard box onto  $[a, b] \times [c, d]$ .

We note that as  $[a, b] \times [c, d]$  runs through all boxes of Liouville measure  $\log 2$ ,  $\gamma_{[a,b] \times [c,d]}$  runs through all Möbius maps of  $\mathbb{D}$ , which implies that the above supremum can be taken over the space  $Mob(\mathbb{D})$  of all Möbius maps that preserve the unit disk  $\mathbb{D}$ . Moreover, since any continuous  $f : [a_0, b_0] \times [c_0, d_0] \rightarrow \mathbb{R}$  with  $\mathcal{L}([a_0, b_0] \times [c_0, d_0]) = \log 2$  can be pulled back to a continuous  $f \circ \gamma_{[a_0, b_0] \times [c_0, d_0]} : [1, i] \times [-1, -i] \rightarrow \mathbb{R}$  and since the above supremum is over all Möbius maps, we do not need to restrict to continuous functions with supports on the standard box, but rather to continuous maps with supports in any box with Liouville measure  $\log 2$ . In addition, if a continuous  $f : \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag} \rightarrow \mathbb{R}$  has a compact support then it can be written as a finite sum of continuous functions with supports in boxes of Liouville measures  $\log 2$ . Therefore

**Definition 5.1.** A sequence of geodesic currents  $\{\alpha_n\}_n$  converges in the *uniform* weak\* topology to  $\alpha$  if for every continuous function  $f : \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag} \rightarrow \mathbb{R}$  with compact support

$$\sup_{\gamma \in Mob(\mathbb{D})} \left| \int_{\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}} f[d\gamma^*(\alpha_n - \alpha)] \right| \rightarrow 0$$

as  $n \rightarrow \infty$ .

This definition is equivalent to the first definition using boxes of Liouville measure  $\log 2$ .

Assume that  $\alpha_n$  converges to  $\alpha$  in the weak\* topology. Below we formulate a sufficient condition guaranteeing that  $\alpha_n$  does not converge to  $\alpha$  in the uniform weak\* topology. Given  $\delta > 0$ , assume that there exist  $C_1, C_2, C_3$  and a sequence of boxes  $Q_k = [a_k, b_k] \times [c_k, d_k]$  and sub-boxes  $Q'_k = [a'_k, b'_k] \times [c'_k, d'_k]$  compactly contained in the interior of  $Q_k$  such that

$$(15) \quad \mathcal{L}(Q_k) \leq C_1,$$

$$(16) \quad \mathcal{L}(Q'_k) \geq C_2 > 0,$$

$$(17) \quad \min\{\mathcal{L}([a_k, a'_k] \times [c_k, d_k]), \mathcal{L}([b'_k, b_k] \times [c_k, d_k]), \\ \mathcal{L}([a_k, b_k] \times [c_k, c'_k]), \mathcal{L}([a_k, b_k] \times [d'_k, d_k])\} \geq \delta > 0,$$

$$(18) \quad \alpha_{n_k}(Q'_k) \geq C_3 > 0,$$

for some  $n_k$  with  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ , and

$$(19) \quad \alpha(Q_k) \rightarrow 0,$$

as  $k \rightarrow \infty$ , where  $C'_1, C''_1, C_2$  and  $C_3$  are independent of  $k$  and  $\delta$ .

We now establish that  $\alpha_n$  does not converge to  $\alpha$  in the uniform weak\* topology if the above conditions are satisfied. Let  $Q = [a, b] \times [c, d]$  be a fixed box with  $\mathcal{L}(Q) = C_1$  and let  $Q' = [a', b'] \times [c', d']$  be a box compactly contained in the interior of  $Q$  such that

$$(20) \quad \mathcal{L}([a, a'] \times [c, d]) = \mathcal{L}([b', b] \times [c, d]) = \\ \mathcal{L}([a, b] \times [c', c]) = \mathcal{L}([a, b] \times [d', d]) = \delta.$$

Let  $\gamma_k \in Mod(\mathbb{D})$  be such that  $\gamma_k(Q) \supseteq Q_k$ . Then (20) and (17) imply that  $\gamma_k(Q') \supseteq Q'_k$ . Let  $f : \mathbb{S}^1 \times \mathbb{S}^1 - \text{diag} \rightarrow \mathbb{R}$  be a continuous functions such that the support of  $f$  is contained in  $Q$ ,  $0 \leq f \leq 1$  and  $f|_{Q'} = 1$ . Then by (18) and (19) we have

$$\int_{\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}} f d\gamma_k^*[\alpha_{n_k} - \alpha] \geq C_3 - \alpha(Q_k) > C_3/2 > 0$$

when  $n_k$  is large, which implies that  $\alpha_{n_k}$  does not converge in the uniform weak\* topology to  $\alpha$ . Since the uniform weak\* convergence implies the weak\* convergence and since  $\alpha_n$  converges in the weak\* topology to  $\alpha$ , it follows that  $\alpha_n$  does not converge to any geodesic current in the uniform weak\* topology.

We find an example of an integrable holomorphic quadratic differential  $\varphi$  on the unit disk  $\mathbb{D}$  such that the corresponding Teichmüller ray  $T_\epsilon$  does not converge in the uniform weak\* topology to  $\mu_\varphi$  while Theorem 4.5 established that it does converge to  $\mu_\varphi$  in the weak\* topology. The differential  $\varphi$  is constructed by taking the pull back of  $dz^2$  on the domain  $D$  in the lemma below under the Riemann mapping. For simplicity of notation, we denote by  $T_\epsilon : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  the boundary map of the Teichmüller geodesic ray  $T_\epsilon$ . The above criterion is used for the family of Liouville currents  $\alpha_\epsilon := \epsilon T_\epsilon^*(\mathcal{L})$  when  $\epsilon \rightarrow 0^+$  and the weak\* limit  $\mu_\varphi$ . The conditions (15), (16), (17), (18) and (19) are replaced by equivalent conditions in terms of the moduli of the families of curves connecting two intervals on  $\mathbb{S}^1$  defining the box of geodesics.

**Lemma 5.2.** *There is a domain  $D \subset \mathbb{C}$  of finite area with the following properties. There exist constants  $0 < C'_1, C'_2, C'_3, \delta' < \infty$ , a sequence of arcs  $[a_k, b_k], [c_k, d_k]$  and sub-arcs  $[a'_k, b'_k], [c'_k, d'_k]$  on  $\mathbb{S}^1$  and a sequence  $\epsilon_k > 0$  approaching 0 such that with notations as above we have*

- (a).  $\text{mod}([a_k, b_k], [c_k, d_k]; \mathbb{D}) \leq C'_1, \forall k \in \mathbb{N}$
- (b).  $\text{mod}([a'_k, b'_k], [c'_k, d'_k]; \mathbb{D}) \geq C'_2, \forall k \in \mathbb{N}$
- (c).  $\min\{\text{mod}([a_k, a'_k], [c_k, d_k]; \mathbb{D}), \text{mod}([b'_k, b_k], [c_k, d_k]; \mathbb{D}), \text{mod}([a_k, b_k], [c_k, c'_k]; \mathbb{D}), \text{mod}([a_k, b_k], [d'_k, d_k]; \mathbb{D})\} \geq \delta' > 0,$
- (d).  $\mu_\varphi([a_k, b_k] \times [c_k, d_k]) \rightarrow 0, \text{ as } k \rightarrow \infty$
- (e).  $\epsilon_k \cdot \text{mod}(T_{\epsilon_k}^\varphi([a'_k, b'_k]), T_{\epsilon_k}^\varphi([c'_k, d'_k]); T_{\epsilon_k}^\varphi(\mathbb{D})) \geq C'_3.$

**Remark 5.3.** We would like to emphasize again that in the lemma above  $\varphi$  denotes the quadratic differential which is the pullback of  $dz^2$  under the Riemann map of  $D$  and  $T_\epsilon^\varphi$  is the corresponding Teichmüller mapping. The boxes  $[a_k, b_k] \times [c_k, d_k]$  and  $[a'_k, b'_k] \times [c'_k, d'_k]$  on  $\mathbb{S}^1$  under the Riemann mapping correspond to boxes in  $D$ , and this correspondence is implicitly assumed.

*Proof of Lemma 5.2.* Below we will define the domain  $D$  as well as a sequence of continua  $E_k, E'_k, F_k, F'_k \subset \partial D$ , which are the preimages of the intervals  $[a_k, b_k], [a'_k, b'_k], [c_k, d_k], [c'_k, d'_k] \subset \mathbb{S}^1$  under the Riemann mapping of  $D$ . In particular  $E'_k \subset E_k$  and  $F'_k \subset F_k$ . Moreover, instead of estimating the moduli of the curve families in the unit disc  $\mathbb{D}$ , we will obtain the estimates in  $D$ . To simplify the notation we let

$$\Gamma_k := (E_k, F_k; D) \text{ and } \Gamma'_k := (E'_k, F'_k; D).$$

Furthermore, denoting the two nonempty components of  $E_k \setminus E'_k$  and  $F_k \setminus F'_k$  by  $E_k^i$  and  $F_k^i$ ,  $i \in \{1, 2\}$ , respectively, we let

$$\Gamma_k^{i,j} := (E_k^i, F_k^j; D).$$

Just as before, given two continua  $E, F \subset \partial D$  we denote by  $\Gamma_v(E, F; D)$  the family of vertical curves connecting  $E$  and  $F$  in  $D$ .

By conformal invariance of the modulus and Theorem 4.2 conditions (a) – (e) are equivalent to the following:

- (a').  $\text{mod} \Gamma_k \leq C'_1, \forall k \in \mathbb{N},$
- (b').  $\text{mod} \Gamma'_k \geq C'_2, \forall k \in \mathbb{N},$
- (c').  $\text{mod} \Gamma_k^{i,j} \geq \delta', \forall k \in \mathbb{N}, \forall i, j \in \{1, 2\},$
- (d').  $\lim_{k \rightarrow \infty} \text{mod} \Gamma_v(E_k, F_k; D) = 0,$
- (e').  $\epsilon_k \cdot \text{mod}(T_{\epsilon_k}(\Gamma'_k)) \geq C'_3, \forall k \in \mathbb{N}.$

Next, we construct the domain  $D$  and prove properties (a') – (e').

For  $k = 1, 2, \dots$ , and  $j = 0, 1, \dots, 2^k$  let

$$L_{k,j} := \left\{ \left( \frac{1}{2^k} + \frac{j}{2^{2k}}, y \right) : \frac{1}{2^k} \leq y \leq 1 \right\}.$$

Define

$$D := [0, 1]^2 \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{2^k} L_{k,j}.$$

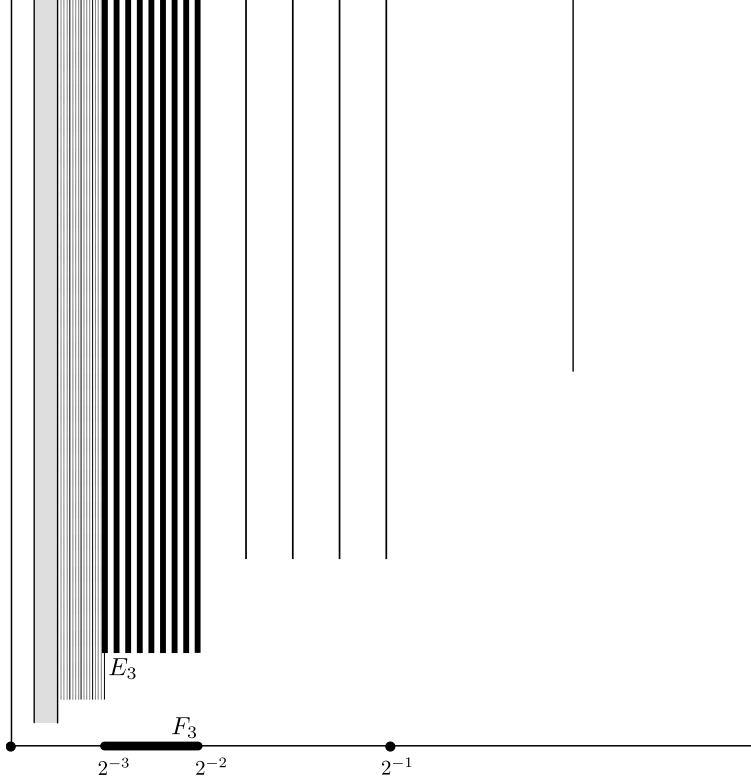


FIGURE 4. The domain  $D$ . The bold interval is  $F_3$ , while the part of  $\partial D$  above it is  $E_3$ .

Now, for  $k \geq 1$  let

$$(21) \quad F_k = \left[ \frac{1}{2^k}, \frac{2}{2^k} \right] \quad \text{and} \quad E_k = \bigcup_{j=0}^{2^k} L_{k,j} \cup \{(x, 1) : x \in F_k\},$$

$$(22) \quad F'_k = \frac{1}{2} F_k \quad \text{and} \quad E'_k = \bigcup_{j=\frac{1}{4}2^k}^{\frac{3}{4}2^k} L_{k,j} \cup \{(x, 1) : x \in F'_k\},$$

where  $\frac{1}{2}F_k$  denotes the interval with the same center as  $F_k$  but half the length.

**Proof of (a').** Since

$$\Delta(E_k, F_k) = \frac{\text{dist}(E_k, F_k)}{\min\{\text{diam} E_k, \text{diam} F_k\}} = \frac{2^{-k}}{2^{-k}} = 1$$

for every  $k \geq 1$ , by Lemma 3.2 we have

$$\text{mod}(\Gamma_k) = \text{mod}(E_k, F_k; D) \leq \text{mod}(E_k, F_k; \mathbb{C}) \leq \frac{9}{4}\pi.$$

**Proof of (b').** To estimate  $\text{mod} \Gamma'_k$  from below we will use conjugate families. Recall that if continua  $E, F \subset \partial D$  then the family of curves separating  $E$  and  $F$  in  $D$  is called the family conjugate to  $(E, F; D)$ . We will denote by  $(E, F; D)^t$  the family conjugate to  $(E, F, D)$ . The modulus of  $(E, F; D)^t$  may be found as follows, see [5]

$$(23) \quad \text{mod}((E, F; D)^t) = \frac{1}{\text{mod}(E, F; D)}.$$

Thus, to estimate  $\text{mod} \Gamma'_k$  from below we can instead estimate  $\text{mod}((\Gamma'_k)^t)$  from above. Note, that every curve  $\gamma \in (\Gamma'_k)^t$  contains a subcurve  $\delta$  connecting the two components of  $\partial D \setminus (F'_k \cup E'_k)$  in the rectangle

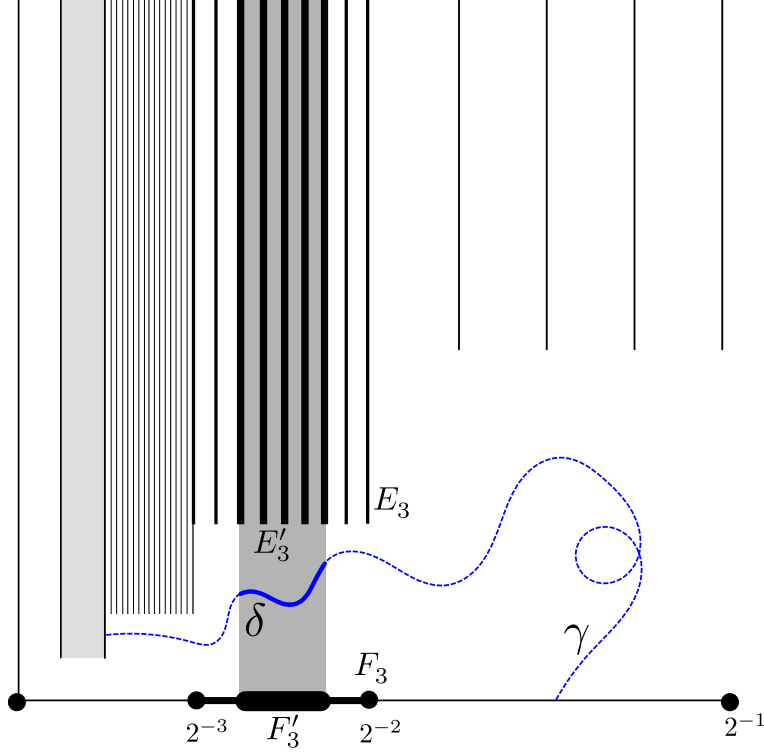


FIGURE 5. Estimating  $\text{mod}(F'_k, E'_k; D)$  from below. Every curve  $\gamma \in (\Gamma'_k)^t$  separating  $E'_k$  from  $F'_k$  in  $D$  contains a subcurve  $\delta \in G_k$  which connects the two components of  $\partial(F'_k \times (0, 1)) \setminus (F'_k \cup E'_k)$  within the grey rectangle  $F'_k \times (0, 1)$ .

$F'_k \times [0, 1]$ , see Fig. 5. Therefore,

$$(24) \quad \text{mod}(\Gamma'_k)^t \leq \text{mod} G_k,$$

where by  $G_k$  we denote the family of curves connecting the two components of  $\partial(F'_k \times (0, 1)) \setminus (F'_k \cup E'_k)$  in the rectangle  $F'_k \times [0, 1]$ . Next we estimate  $\text{mod} G_k$  using the following result.

**Lemma 5.4.** *Let  $0 < a < b < 1$ ,  $0 < c < b$  and  $N \geq 1$ . Denote by  $D_N$  the domain*

$$D_N = (0, a) \times (0, b) \setminus \bigcup_{i=0}^N \{x_i\} \times [c, 1],$$

where  $x_i = \frac{ai}{N}$  and by  $\Gamma_N$  the family of curves in  $D_N$  connecting the vertical intervals  $(0, ic)$  to  $(a, a + ic)$  (see Figure 5). Then

$$(25) \quad \frac{c}{a} \leq \text{mod} \Gamma_N \leq \frac{c}{a} + \frac{1}{2N}.$$

*Proof.* The first estimate follows from the fact that  $\Gamma_N$  contains the family connecting the vertical sides in the rectangle  $[0, a] \times [0, c]$ . To obtain the upper bound consider the rectangle  $R_N = (0, a) \times (0, c + \frac{a}{2N})$  and define

$$\rho_N = \frac{1}{a} \chi_{R_N \cap D_N}.$$

We next show that  $\rho_N$  is admissible for  $\Gamma_N$ . For that, let  $\gamma \in \Gamma_N$  and let (cf. Figure 6)

$$\gamma_i = \gamma \cap ((x_i, x_{i+1}) \times (0, 1)), \quad i = 0, \dots, N-1.$$

Next, we show that  $l(\gamma_i \cap R_N) \geq x_{i+1} - x_i = \frac{a}{N}$ . Indeed, if  $\gamma_i \cap \partial R_N = \emptyset$  then, since  $\gamma_i \cap R_N$  is a connected curve connecting the vertical sides of the rectangle  $R_N \cap ((x_i, x_{i+1}) \times (0, 1))$ , we have  $l(\gamma_i \cap R_N) \geq a/N$ .



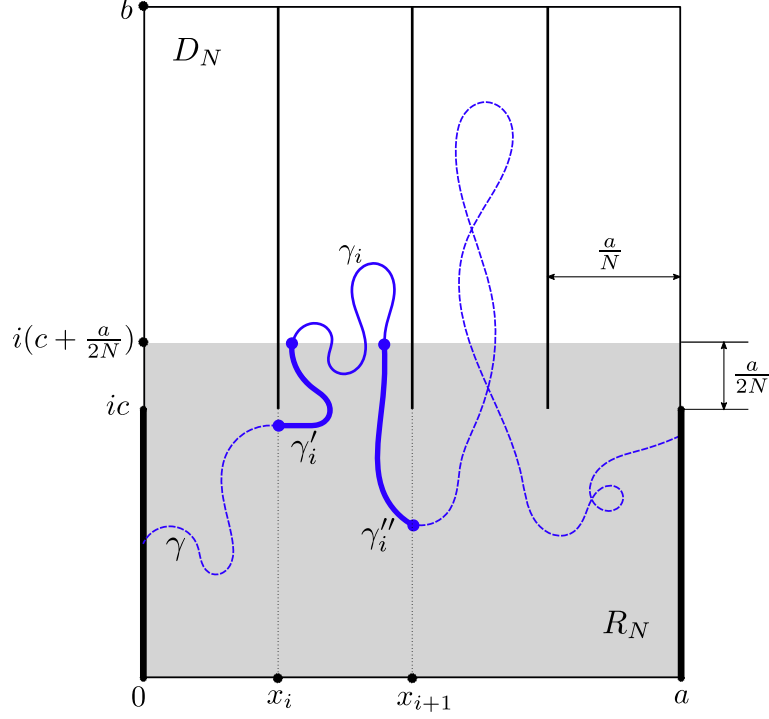


FIGURE 6.

On the other hand if  $\gamma_i \cap \partial R_N \neq \emptyset$  then there are two connected components  $\gamma'_i, \gamma''_i$  of  $\gamma_i$  which connect the vertical intervals  $\{x_i\} \times (0, c)$  and  $\{x_{i+1}\} \times (0, c)$  to the horizontal interval  $(x_i, x_{i+1}) \times \{c + \frac{a}{2N}\}$  in  $D_N$ , respectively. Since the distance between the aforementioned vertical intervals and the horizontal interval is at least  $a/(2N)$  we obtain

$$l(\gamma_i \cap R_N) \geq l(\gamma'_i) + l(\gamma''_i) \geq 2 \frac{a}{2N} = \frac{a}{N}.$$

Thus we have

$$l_{\rho_N}(\gamma) \geq \sum_{i=0}^{N-1} l_{\rho_N}(\gamma_i) = \frac{1}{a} \sum_{i=0}^{N-1} l(\gamma_i \cap R_N) \geq \frac{1}{a} \cdot N \cdot \frac{a}{N} = 1,$$

and  $\rho_N$  is admissible for  $\Gamma_N$ . Therefore we can estimate the modulus of  $\Gamma_N$  as follows.

$$\text{mod} \Gamma_N \leq \int \rho_N^2 = \frac{1}{a^2} |R_N \cap D_N| = \frac{1}{a^2} \cdot a(c + \frac{a}{2N}) = \frac{c}{a} + \frac{1}{2N}. \quad \square$$

Using the lemma we see that

$$\text{mod} \Gamma'_k \geq \frac{1}{\text{mod}((\Gamma'_k)^t)} \geq \frac{1}{\text{mod} G_k} \geq \frac{1}{\frac{2^{-k}}{2^{-k/2}} + \frac{1}{2 \cdot 2^k}} > \frac{1}{3},$$

for  $k \geq 1$ . Which proves (b').

**Proof of (c').** We start by estimating the modulus of  $\Gamma_k^{1,1} = (E_k^1, F_k^1; D)$ . Note that

$$F_k^1 = \left( \frac{1}{2^k}, \frac{1+1/4}{2^k} \right)$$

$$F_k^2 = \left( \frac{1+3/4}{2^k}, \frac{2}{2^k} \right),$$

while  $E_k^1$  and  $E_k^2$  are the parts of  $E_k$  above  $F_k^1$  and  $F_k^2$ , respectively.

Just like in the proof of (b') we use Lemma 5.4 to obtain the following estimate

$$(26) \quad \begin{aligned} \text{mod}(F_k^1, E_k^1, D) &\geq \text{mod}(F_k^1, E_k^1, F_k^1 \times [0, 1]) \\ &= \frac{1}{\text{mod}(F_k^1, E_k^1, F_k^1 \times [0, 1])^t} \geq \frac{1}{\frac{2^{-k}}{2^{-k/4}} + \frac{1}{2(2^{k/4})}} \geq \frac{1}{5}, \end{aligned}$$

for  $k \geq 1$ . The same way we also obtain  $\text{mod}(F_k^2, E_k^2, D) \geq 1/5$ . Next, we estimate  $\text{mod}(F_k^1, E_k^2, D)$  as follows

$$(27) \quad \text{mod}(F_k^1, E_k^2, D) \geq \text{mod}(F_k^1, E_k^2, F_k \times [0, 1]) = \frac{1}{\text{mod}(F_k^1, E_k^2, F_k^1 \times [0, 1])^t}.$$

But

$$(F_k^1, E_k^2, F_k^1 \times [0, 1])^t = (F_k'', E_k'', F_k^1 \times [0, 1])$$

where  $E_k''$  and  $F_k''$  are the two components of  $\partial(F_k \times (0, 1)) \setminus (F_k \cup E_k)$ , or

$$\begin{aligned} E_k'' &= [2^{-k}, (1+i)2^{-k}] \cup E_k^1 \cup E_k', \\ F_k'' &= [2^{-k+1}, (1+i)2^{-k+1}] \cup F_k' \cup F_k^2. \end{aligned}$$

Since  $\text{dist}(E_k'', F_k'') = \text{diam} F_k^1 = 2^k/4$  and  $\text{diam} E_k'' \geq \text{diam} F_k''$  we have

$$\Delta(E_k'', F_k'') = \frac{\text{diam} F_k^1}{\text{diam} F_k''} \geq \frac{2^{-k}/4}{2 \cdot 2^{-k}} = \frac{1}{8}.$$

Therefore by Lemma 3.2 we have

$$\text{mod}(F_k'', E_k'', F_k^1 \times [0, 1]) \leq \pi(1+4)^2 = 25\pi,$$

and we finally obtain

$$\text{mod}(F_k^1, E_k^2, D) \geq \frac{1}{25\pi}.$$

In the same way we can show that  $\text{mod}(F_k^2, E_k^1, D) \geq \frac{1}{25\pi}$  and thus prove (c').

**Proof of (d').** As was shown before we have,

$$\text{mod} \Gamma_v(E_k, F_k; D) = \int_{F_k} \frac{dx}{l(x)}$$

where in this case  $l(x)$  is the Euclidean length of the vertical trajectory passing through  $x \in \mathbb{C}$  and integration is with respect to the Lebesgue measure. Thus, since  $l(x) = 1$  for almost every  $x \in F_k$  we obtain

$$\text{mod} \Gamma_v(E_k, F_k; D) = |F_k| = \frac{1}{2^k} \rightarrow 0.$$

**Proof of (e').** Just like above, let  $G_k$  be the family of curves connecting the two vertical intervals in  $\partial D$  (namely, the two components of the boundary of  $\partial(F_k' \times (0, 1)) \setminus (F_k' \cup E_k')$ ). Then  $(\Gamma_k')^t$  overflows  $G_k$ , and the same way we also have  $T_\varepsilon((\Gamma_k')^t)$  overflows  $T_\varepsilon(G_k)$  and therefore for every  $\varepsilon > 0$  we have

$$\text{mod} T_\varepsilon((\Gamma_k')^t) \leq \text{mod} T_\varepsilon((G_k')^t).$$

Next, we let  $\varepsilon_k = 2^{-k}$  and estimate  $\text{mod} T_{\varepsilon_k}(G_k)$  from above. Considering the conformal mapping

$$f_k(z) = \frac{1}{\varepsilon_k} \left( z - \frac{1}{2^k} \right)$$

and using Lemma 5.4 we obtain that

$$\text{mod} T_{\varepsilon_k}(G_k) = \text{mod} T_{\varepsilon_k}(f_k(G_k)) \leq \frac{\varepsilon_k}{1/2} + \frac{1}{2 \cdot (2^k/2)} = 3\varepsilon_k.$$

Now, since  $T_{\varepsilon_k}(G_k) = (T_{\varepsilon_k}(G_k'))^t$ , we obtain

$$\varepsilon_k \text{mod}(T_{\varepsilon_k}(\Gamma_k')) = \varepsilon_k \cdot \frac{1}{\text{mod}(T_{\varepsilon_k}((\Gamma_k')^t))} \geq \frac{\varepsilon_k}{\text{mod}(T_{\varepsilon_k}(G_k))} \geq \frac{\varepsilon_k}{3\varepsilon_k} = \frac{1}{3}.$$

□

## 6. FROM INTEGRABLE HOLOMORPHIC QUADRATIC DIFFERENTIALS TO BOUNDED MEASURED LAMINATIONS

Let  $\varphi$  be an integrable holomorphic quadratic differential on  $\mathbb{D}$  (i.e. a holomorphic function  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$  such that  $\|\varphi\|_{L^1(\mathbb{D})} = \iint_{\mathbb{D}} |\varphi(z)| dx dy < \infty$ ). Let  $PA(\mathbb{D})$  be the space of all integrable holomorphic quadratic differentials on  $\mathbb{D}$ .

Given  $\varphi \in A(\mathbb{D})$ , we defined a corresponding bounded measured lamination

$$\mu_{\varphi}([a, b] \times [c, d]) = \text{mod}(\Gamma_v([a, b], [c, d]))$$

or equivalently

$$\mu_{\varphi}([a, b] \times [c, d]) = \int_I \frac{1}{l(z)} |\sqrt{\varphi(z)} dz|$$

where  $I$  is transverse arc to  $\Gamma_v([a, b], [c, d])$ .

It follows that if  $c > 0$  then  $\mu_{c\varphi} = \mu_{\varphi}$ . Therefore we obtain a map from the space  $PA(\mathbb{D})$  of projective integrable holomorphic quadratic differentials to the space of projective bounded measured laminations  $PML_{bdd}(\mathbb{D})$ ,

$$\mathcal{M} : PA(\mathbb{D}) \rightarrow PML_{bdd}(\mathbb{D}).$$

We prove that  $\mathcal{M} : PA(\mathbb{D}) \rightarrow PML_{bdd}(\mathbb{D})$  is injective.

**Theorem 6.1.** *The map*

$$\mathcal{M} : PA(\mathbb{D}) \rightarrow PML_{bdd}(\mathbb{D}).$$

*defined by*

$$\mathcal{M}([\varphi]) = [\mu_{\varphi}]$$

*is injective.*

*Proof.* We assume that

$$(28) \quad \mu_{\varphi} = c_1 \mu_{\varphi'}$$

and need to prove that  $\varphi = c\varphi'$  for some  $c > 0$ . Since  $\mu_{\varphi} = c_1 \mu_{\varphi'}$  we have that their geodesic laminations supports  $|\mu_{\varphi}|$  and  $|c_1 \mu_{\varphi'}|$  are the same. In other words each leaf of the vertical foliation  $\varphi$  is homotopic to a leaf of the vertical foliation of  $\varphi'$  relative their two endpoints on the unit circle, and vice versa.

Additionally, assume that the corresponding leaves of the vertical foliations are not only homotopic but that they are equal to each other. In other words, the vertical foliations of  $\varphi$  and  $\varphi'$  are equal. If  $z_0 \in \mathbb{D}$  is a regular point of both  $\varphi$  and  $\varphi'$ , denote by  $\zeta$  and  $\zeta'$  the corresponding natural parameters in a regular neighborhood  $U$  of  $z_0$ . Then  $f = \zeta' \circ \zeta^{-1}$  is a conformal mapping from  $\zeta(U) \subset \mathbb{C}$  onto  $\zeta'(U) \subset \mathbb{C}$  that maps vertical lines onto vertical lines. It follows then that  $\zeta' = a\zeta + b$  for some  $a \in \mathbb{R}$ . Thus  $d\zeta'^2 = a^2 d\zeta^2$  and we set  $c = a^2$ .

We obtained that for each regular point  $z_0$  of  $\varphi$  and  $\varphi'$  there exist a neighborhood  $U \ni z_0$  and a constant  $c = a^2 > 0$  such that  $\varphi = c\varphi'$  in  $U$ . Since the set of regular points of  $\varphi$  and  $\varphi'$  is connected and dense in  $\mathbb{D}$  then  $\varphi = c\varphi'$  in  $\mathbb{D}$  and the proof is finished in this case.

It remains to prove that the vertical foliations of  $\varphi$  and  $\varphi'$  are the same under the assumption that  $\mu_{\varphi} = c_1 \mu_{\varphi'}$ . Let  $\{S(\beta_i)\}_{i=1}^{\infty}$  be a family of mutually disjoint vertical strips with open transverse horizontal arcs  $\beta_i$  that covers  $\mathbb{D}$  up to countably many vertical trajectories (cf. [21]). The metric on the horizontal arcs  $\beta_i$  is induced by  $|\sqrt{\varphi(z)} dz|$  and we isometrically identify  $\beta_i$  with  $(0, a_i)$ , where  $a_i$  is the length of  $\beta_i$ . The variable in  $(0, a_i)$  is  $x$  and the integration with respect  $dx$  corresponds to integration with respect  $|\sqrt{\varphi(z)} dz|$  in  $\mathbb{D}$ . The arc  $(0, a_i)$  is a horizontal arc in the natural parameter  $w = \int \sqrt{\varphi(z)} dz$  for  $\varphi(z)$ .

For  $\beta_i$ , let  $S(\beta_i, (0, x))$  be the substrip of  $S(\beta_i)$  of vertical trajectories going through  $(0, x) \subset (0, a_i)$ . The area of  $S(\beta_i, (0, x))$  is

$$A_{\beta_i}^{\varphi}(x) = \int_{(0, x)} l^{\varphi}(v_{\beta_i}^{\varphi}(t)) dt,$$

where  $v_{\beta_i}^{\varphi}(t)$  is the vertical trajectory of  $\varphi$  through the point  $t \in (0, x) \subset \beta_i$  and  $l^{\varphi}(\cdot)$  is the length in the  $|\sqrt{\varphi(z)} dz|$  metric. The modulus of the vertical trajectories in  $S(\beta_i, (0, x))$  is

$$M_{\beta_i}^{\varphi}(x) = \int_{(0, x)} \frac{1}{l^{\varphi}(v_{\beta_i}^{\varphi}(t))} dt.$$

If necessary, we multiply  $\varphi'$  by a positive constant such that  $\|c_1\varphi'\|_{L^1} = \|\varphi\|_{L^1}$ . Since the supports of  $\mu_\varphi$  and  $\mu_{\varphi'}$  are the same, to each  $S(\beta_i, (0, x))$  there corresponds a vertical strip  $\tilde{S}(\beta_i, (0, x))$  of vertical trajectories  $v_{\beta_i}^{\varphi'}(t)$  of  $\varphi'$  with the same endpoints on  $\mathbb{S}^1$  as  $v_{\beta_i}^\varphi(t)$ . Note that  $v_{\beta_i}^{\varphi'}(t)$  does not necessarily pass through  $t \in \beta_i$  or even intersects  $\beta_i$ .

Let  $A_{\beta_i}^{\varphi'}(x)$  and  $M_{\beta_i}^{\varphi'}(x)$  denote the area of  $\tilde{S}(\beta_i, (0, x))$  and the modulus of vertical trajectories of  $\varphi'$  in  $\tilde{S}(\beta_i, (0, x))$ . We have the following lemma.

**Lemma 6.2.** *Let  $\beta_i$  be a transverse horizontal arc to a vertical strip  $S(\beta_i)$  isometrically identified with  $(0, a_i)$  in the natural parameter of  $\varphi$ . Then for a.e.  $x \in (0, a_i)$ , we have*

$$\frac{d}{dx} M_{\beta_i}^{\varphi'}(x) \leq \frac{\frac{d}{dx} A_{\beta_i}^{\varphi'}(x)}{[l^\varphi(v_{\beta_i}^{\varphi'}(x))]^2},$$

where  $l^\varphi(\cdot)$  is the  $\varphi$ -length and  $v_{\beta_i}^{\varphi'}(x)$  is the horizontal trajectory of  $\varphi'$  whose endpoints agree with the endpoints of  $v_{\beta_i}^\varphi(x)$ .

*Proof.* For  $x \in (0, a_i)$  and small  $\varepsilon > 0$  we denote

$$L_x(\varepsilon) = \inf\{l^\varphi(v_{\beta_i}^{\varphi'}(t)) : t \in [x, x + \varepsilon]\}.$$

Note that  $L_x(\varepsilon) > 0$  for  $\varepsilon > 0$  small enough by the continuity of  $\varphi'$ . It is possible that  $L_x(\varepsilon) = \infty$  if all vertical trajectories of  $\varphi'$  close to  $v_{\beta_i}^{\varphi'}(x)$  have infinite  $\varphi$ -length. The metric  $\rho(z) \equiv L_x(\varepsilon)^{-1}$  is admissible for  $\tilde{S}(\beta_i, [x, x + \varepsilon])$ , where  $L_x(\varepsilon)^{-1} = 0$  if  $L_x(\varepsilon) = \infty$ .

Since,  $L_x(\varepsilon)$  is non-increasing it has a limit as  $\varepsilon \rightarrow 0^+$ . In fact, we have

$$L_x(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} l^\varphi(v_{\beta_i}^{\varphi'}(t)).$$

To see this, note first that  $L_x(\varepsilon) \leq l^\varphi(v_{\beta_i}^{\varphi'}(x))$  and we only need to estimate the limit from below.

Assume first that  $l^\varphi(v_{\beta_i}^{\varphi'}(x)) < \infty$ . Fix  $\delta > 0$  and choose points  $\xi_0, \dots, \xi_k \in v_{\beta_i}^{\varphi'}(x)$ , so that

$$(29) \quad \sum_{i=1}^k d_\varphi(\xi_i, \xi_{i-1}) \geq l^\varphi(v_{\beta_i}^{\varphi'}(x)) - \frac{\delta}{2}$$

where  $d_\varphi(\xi_i, \xi_{i-1})$  is the distance between  $\xi_i$  and  $\xi_{i-1}$  in the  $\varphi$  metric, i.e. the metric induced by  $|\sqrt{\varphi(z)}dz|$ .

We want to show that for small  $\eta$  the curves  $v_{\beta_i}^{\varphi'}(x + \eta)$  have lengths at least  $l^\varphi(v_{\beta_i}^{\varphi'}(x)) - \delta$ . Since the set of vertical trajectories  $S(\beta_i)$  foliates a neighborhood of  $v_{\beta_i}^\varphi(x)$  then  $\tilde{S}(\beta_i)$  must foliate a neighborhood of  $v_{\beta_i}^{\varphi'}(x)$  because the separation property of the vertical trajectories of  $\varphi$  and  $\varphi'$  is a topological property of their endpoints. By choosing small  $\eta > 0$ , we get that a subarc of  $v_{\beta_i}^{\varphi'}(x + \eta)$  is within small euclidean distance to the subarc of  $v_{\beta_i}^{\varphi'}(x)$  between  $\xi_0$  and  $\xi_k$  for all  $0 < \eta < \eta$ . Since  $\varphi$  is continuous, it follows that for  $\eta > 0$  small enough, each  $v_{\beta_i}^{\varphi'}(x + \eta)$  for  $\eta < \eta$  has points  $\xi'_0, \dots, \xi'_k$  on the  $\varphi$ -distance less than  $\frac{\delta}{4k}$  from  $\xi_0, \dots, \xi_k$ , respectively. Therefore by (29) we have

$$l^\varphi(v_{\beta_i}^{\varphi'}(x + \eta)) \geq \sum_{i=1}^k d_\varphi(\xi'_i, \xi'_{i-1}) \geq \sum_{i=1}^k \left( d_\varphi(\xi_i, \xi_{i-1}) - \frac{\delta}{2k} \right) \geq l^\varphi(v_{\beta_i}^{\varphi'}(x)) - \delta.$$

Thus  $L_x(\eta) \geq l^\varphi(v_{\beta_i}^{\varphi'}(x)) - \delta$  for all  $\eta < \eta$  which implies that  $\lim_{\eta \rightarrow 0^+} L_x(\eta) = l^\varphi(v_{\beta_i}^{\varphi'}(x))$  because  $\delta > 0$  is arbitrary.

Assume now that  $l^\varphi(v_{\beta_i}^{\varphi'}(x)) = \infty$ . If  $L_x(\varepsilon) = \infty$  for some  $\varepsilon > 0$  then  $\lim_{\varepsilon \rightarrow 0^+} L_x(\varepsilon) = l^\varphi(v_{\beta_i}^{\varphi'}(x))$ . We consider the case when  $L_x(\varepsilon) < \infty$  for all  $\varepsilon > 0$  and need to prove that for every  $M > 0$  there exist  $\eta > 0$  such that  $L_x(\varepsilon) \geq M$  for all  $\varepsilon < \eta$ . Choose points  $\xi_0, \dots, \xi_k \in v_{\beta_i}^{\varphi'}(x)$ , so that

$$(30) \quad \sum_{i=1}^k d_\varphi(\xi_i, \xi_{i-1}) \geq M + 1.$$

where  $d_\varphi(\xi_i, \xi_{i-1})$  is the distance between  $\xi_i$  and  $\xi_{i-1}$  in the  $\varphi$ -metric. For small  $\eta > 0$  and all  $\varepsilon < \eta$ , a subarc of  $v_{\beta_i}^{\varphi'}(x + \varepsilon)$  is within small euclidean distance to the subarc of  $v_{\beta_i}^{\varphi'}(x)$  between  $\xi_0$  and  $\xi_k$ . Since  $\varphi$  is continuous, it follows that for  $\eta > 0$  small enough, each  $v_{\beta_i}^{\varphi'}(x + \varepsilon)$  for  $\varepsilon < \eta$  has points  $\xi'_0, \dots, \xi'_k$  on the  $\varphi$ -distance less than  $\frac{1}{2k}$  from  $\xi_0, \dots, \xi_k$ , respectively. Therefore by (30) we have

$$l^\varphi(v_{\beta_i}^{\varphi'}(x + \eta)) \geq \sum_{i=1}^k \left( d_\varphi(\xi_i, \xi_{i-1}) - \frac{1}{4k} \right) \geq l^\varphi(v_{\beta_i}^{\varphi'}(x)) - 1 \geq M$$

and then  $\lim_{\varepsilon \rightarrow 0} L_x(\varepsilon) = l^\varphi(v_{\beta_i}^{\varphi'}(x))$ .

Thus for a.e.  $x \in (0, a_i)$  we have

$$\begin{aligned} \frac{d}{dx} M_{\beta_i}^{\varphi'}(x) &= \lim_{\varepsilon \rightarrow 0^+} \frac{\text{mod} \tilde{S}(\beta_i, [x, x + \varepsilon])}{\varepsilon} \\ &\leq \limsup_{\varepsilon \rightarrow 0^+} \frac{A_{\beta_i}^{\varphi'}(x + \varepsilon) - A_{\beta_i}^{\varphi'}(x)}{\varepsilon L_x^2(\varepsilon)} = \frac{\frac{d}{dx} A_{\beta_i}^{\varphi'}(x)}{[l^\varphi(v_{\beta_i}^{\varphi'}(x))]^2}, \end{aligned}$$

and the proof is complete.  $\square$

Using the above lemma we establish the next lemma which finishes the proof.

**Lemma 6.3.** *If for every  $\beta_i$  and a.e.  $x \in (0, a_i)$  we have  $M_{\beta_i}^\varphi(x) = c_1 M_{\beta_i}^{\varphi'}(x)$  then the vertical foliations of  $\varphi$  and  $\varphi'$  are equal.*

*Proof.* Note that  $M_{\beta_i}^\varphi(x) = \mu_\varphi((0, x))$  and  $M_{\beta_i}^{\varphi'}(x) = \mu_{\varphi'}((0, x))$ , and equation (28) imply that

$$M_{\beta_i}^\varphi(x) = c_1 M_{\beta_i}^{\varphi'}(x)$$

for every  $x \in (0, a_i)$ . By the previous lemma, by Lebesgue's differentiation theorem and by absolute continuity of  $M_{\beta_i}^\varphi(x) = \int_{(0, x)} \frac{1}{l^\varphi(v_{\beta_i}^\varphi(t))} dt$  we have for a.e.  $x \in (0, a_i)$

$$\begin{aligned} \frac{\frac{d}{dx} A_{\beta_i}^\varphi(x)}{[l^\varphi(v_{\beta_i}^\varphi(x))]^2} &= \frac{1}{[l^\varphi(v_{\beta_i}^\varphi(x))]^2} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} l^\varphi(v_{\beta_i}^\varphi(t)) dt = \\ (31) \quad \frac{1}{l^\varphi(v_{\beta_i}^\varphi(x))} &= \frac{d}{dx} M_{\beta_i}^\varphi(x) = c_1 \frac{d}{dx} M_{\beta_i}^{\varphi'}(x) \leq \frac{c_1 \frac{d}{dx} A_{\beta_i}^{\varphi'}(x)}{[l^\varphi(v_{\beta_i}^{\varphi'}(x))]^2}. \end{aligned}$$

Since  $l^\varphi(v_{\beta_i}^\varphi(x)) \leq l^\varphi(v_{\beta_i}^{\varphi'}(x))$  with equality implying that the two curves are the same, it follows from (31) that for a.e.  $x \in (0, a_i)$  we have

$$(32) \quad \frac{d}{dx} A_{\beta_i}^\varphi(x) \leq c_1 \frac{d}{dx} A_{\beta_i}^{\varphi'}(x).$$

Note that  $A_{\beta_i}^\varphi(x)$  is absolutely continuous in  $x$ , since  $A_{\beta_i}^\varphi(x) = \int_{(0, x)} l^\varphi(v_{\beta_i}^\varphi(t)) dt$ . Thus

$$A_{\beta_i}^\varphi(x) = \int_0^x \frac{d}{dt} A_{\beta_i}^\varphi(t) dt \leq \int_0^x c_1 \frac{d}{dt} A_{\beta_i}^{\varphi'}(t) dt \leq c_1 A_{\beta_i}^{\varphi'}(x)$$

for a.e.  $x \in (0, a_i)$ . Since

$$\|\varphi\|_{L^1} = \sum_i A_{\beta_i}^\varphi(a_i) \leq c_1 \sum_i A_{\beta_i}^{\varphi'}(a_i) = \|c_1 \varphi'\|_{L^1}$$

and  $\|\varphi\|_{L^1} = \|c_1 \varphi'\|_{L^1}$ , we necessarily have equality for each term  $\beta_i$  and for a.e.  $x \in (0, a_i)$ .

Thus  $A_{\beta_i}^\varphi(x) = c_1 A_{\beta_i}^{\varphi'}(x)$  for a.e.  $x \in (0, a_i)$  which implies  $\frac{d}{dx} A_{\beta_i}^\varphi(x) = c_1 \frac{d}{dx} A_{\beta_i}^{\varphi'}(x)$  for a.e.  $x \in (0, a_i)$ .

Therefore, by (31) we have

$$\frac{\frac{d}{dx} A_{\beta_i}^\varphi(x)}{[l^\varphi(v_{\beta_i}^\varphi(x))]^2} \leq \frac{c_1 \frac{d}{dx} A_{\beta_i}^{\varphi'}(x)}{[l^\varphi(v_{\beta_i}^{\varphi'}(x))]^2},$$

where the numerators are equal for a.e.  $x \in (0, a)$ . In particular,  $l^\varphi(v_{\beta_i}^{\varphi'}(x)) \leq l^\varphi(v_{\beta_i}^\varphi(x))$  and thus  $l^\varphi(v_{\beta_i}^{\varphi'}(x)) = l^\varphi(v_{\beta_i}^\varphi(x))$  for a.e.  $x$ . By the uniqueness of geodesics in the  $\varphi$  metric connecting two boundary points of simply connected domains (cf. [21]) and since vertical trajectories foliate  $\mathbb{D}$ , we obtain that all vertical trajectories of  $\varphi$  and  $\varphi'$  are the same.  $\square$

The above lemma together with the above finishes the proof of the theorem.  $\square$

Given an integrable holomorphic quadratic differential  $\varphi$  on the unit disk, we denote by  $\nu_\varphi$  the measured lamination whose support  $v_\varphi$  is homotopic to the leaves of the vertical foliation of  $\varphi$  and the transverse measure is given by  $\int_I |\sqrt{\varphi(z)} dz|$ , where  $I$  is a horizontal arc intersecting the leaves of the vertical foliation corresponding to the leaves of  $\nu_\varphi$ . We prove that  $\nu_\varphi$  is Thurston bounded.

**Proposition 6.4.** *Let  $\varphi$  be an integrable holomorphic quadratic differential on the unit disk  $\mathbb{D}$ . Then the vertical foliation measure  $\nu_\varphi$  defined above is Thurston bounded.*

*Proof.* Let  $\mathcal{V}^{\geq 1}$  be the set of all vertical trajectories of  $\varphi$  whose  $\varphi$ -length is  $\geq 1$ . Let  $\mathcal{V}^{< 1}$  be the set of all vertical trajectories of  $\varphi$  whose  $\varphi$ -length is  $< 1$ . Let  $\mathbb{D}^{\geq 1} = \cup_{\gamma \in \mathcal{V}^{\geq 1}} \gamma$  and  $\mathbb{D}^{< 1} = \cup_{\gamma \in \mathcal{V}^{< 1}} \gamma$ .

Let  $[a, b] \times [c, d] \subset (\mathbb{S}^1 \times \mathbb{S}^1) - \text{diag}$  be a box of geodesics with  $\mathcal{L}([a, b] \times [c, d]) = \log 2$ . This implies that  $\frac{1}{C} \leq \text{mod}([a, b], [c, d]; \mathbb{D}) \leq C$  for some  $C > 1$ . Let  $I$  be at most countable union of horizontal arcs of  $\varphi$  that intersects each vertical trajectory of  $\varphi$  in exactly one point. Then we have

$$\|\varphi\|_{L^1} > \iint_{\mathbb{D}^{\geq 1}} |\varphi(z)| dx dy = \int_{I \cap \mathbb{D}^{\geq 1}} l^\varphi(v^\varphi(z)) dx \geq \int_{I \cap \mathbb{D}^{\geq 1}} dx,$$

where  $x$  is the real part of the natural parameter along  $I$ .

For a box of geodesics  $[a, b] \times [c, d]$ , let  $\mathcal{V}_{[a, b] \times [c, d]}^{< 1}$  be the set of vertical trajectories of length less than 1 that connects  $[a, b]$  and  $[c, d]$ . Let  $I_{[a, b] \times [c, d]}^{< 1}$  be the subset of  $I$  that intersects only vertical trajectories of  $\mathcal{V}_{[a, b] \times [c, d]}^{< 1}$ . Then we have

$$C \geq \text{mod}([a, b], [c, d]; \mathbb{D}) \geq \text{mod}(\mathcal{V}_{[a, b] \times [c, d]}^{< 1}) = \int_{I_{[a, b] \times [c, d]}^{< 1}} \frac{1}{l^\varphi(v^\varphi(z))} dx \geq \int_{I_{[a, b] \times [c, d]}^{< 1}} dx.$$

Let  $I_{[a, b] \times [c, d]}$  be the subset of  $I$  that intersects only the vertical trajectories of  $\varphi$  that connect  $[a, b]$  to  $[c, d]$ . Since  $\nu_\varphi([a, b] \times [c, d]) = \int_{I_{[a, b] \times [c, d]}} dx \leq \|\varphi\|_{L^1} + C$ , we have that  $\|\nu_\varphi\|_{Th} < \infty$ .  $\square$

We can recover the integral  $\|\varphi\|_{L^1}$  from  $\mu_\varphi$  and  $\nu_\varphi$  by the formula

$$\|\varphi\|_{L^1} = \int_{\mathbb{S}^1 \times \mathbb{S}^1 - \text{diag}} \frac{d\mu_\varphi}{d\nu_\varphi} d\mu_\varphi.$$

This immediately gives

**Theorem 6.5.** *The map from  $A(\mathbb{D})$  in  $ML_{bdd}(\mathbb{D}) \times ML_{bdd}(\mathbb{D})$  given by*

$$\varphi \mapsto (\nu_\varphi, \mu_\varphi)$$

*is injective.*

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